UNIVERSITY OF ECONOMICS PRAGUE
Faculty of Informatics and Statistics

Management Science

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To the memory of Eva

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1. Introduction

Management is a process used to achieve certain goals through the utilization of resources (people, money, energy, materials, space, time). **Management Science** (MS), an approach to managerial *decision making* based on the scientific method, makes extensive use of quantitative analysis. The alternative name for quantitative approaches to decision making is **Operations Research** (OR). We shall treat both the terms as synonyms throughout the text. Other names for more or less the same area are: operational research, operations analysis, quantitative analysis, quantitative methods, decision analysis and decision science.

The significant development of the Operations Research disciplines and techniques started during World War II in form of military applications (strategic and tactical tasks). After this period many more methodological developments followed and at the end of 20^{th} century the information technology explosion created new possibilities for management science. It is nowadays hardly realizable to carry out an analysis and make decisions without computers. Many software products are used in management science. Some simple decisions can be made using standard spreadsheets (*MS Excel*), whereas the complex real problems require professional software (*Lindo*, *Lingo*, *Xpress*, *AIMMS*, *CPLEX*, *etc*.).

1.1 Management Science – Definition and Characteristics

Two *classical definitions* of management science (operations research), according to Turban and Meredith [8] are:

1. MS/OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problems.

2. MS/OR is the application of the scientific method to the study of the operations of large, complex organizations or activities.

Management science can be briefly defined as: ***The application of the scientific method to the analysis and solution of managerial decision problems***. The major *characteristics* of management science are:

1. A primary focus on managerial *decision making*.
2. The application of the *scientific approach* to the decision making process.
3. The examination of the decision situation from a broad perspective; i.e. the application of a *system approach*.
4. The use of methods and knowledge from *several disciplines*.
5. A reliance on formal *mathematical models*.
6. The extensive use of *computers*.

Decision making is a process of choosing between two or more available alternative courses of action for the purpose of attaining a specific goal. The conclusion of the process is a decision. Decision making is a systematic process and can be simply described by the following steps:

a) Defining the problem.
b) Searching for alternative courses of action.
c) Evaluating the alternatives.
d) Selecting one alternative.

This process is, of course, just one of many possibilities how to approach the problem and is typical for such discipline as **Multicriteria Decision Making**, etc.

The structure of **systems** consists of three related parts: *inputs*, *processes* and *outputs*. *Inputs* enter the system (people, raw materials, money). *Processes* convert the inputs into outputs (processes may use energy, operating procedures, employees, machines). *Outputs* leave the system (products, served customers). In many applications the outputs are connected with the goal of the management (profit maximization, total costs minimization). We consider *closed* and *open systems*. A *closed system* is totally independent of and fully isolated from influences of the *environment* (elements outside the system). An *open system* is significantly interconnected with the environment; this type of system exchanges information, material, energy, people with the environment.

Systems approach involves finding and defining all of the mentioned elements and relations between them in the real world. This approach is the necessary assumption for using management science techniques, as they require the exact description of a problem. Only system approach enables the formulation of the problem in the mathematical way.

1.2 Models and Modeling

The **model** is a fundamental term of management science. **Modeling** is the process of handling real-world problems and describing them in mathematical terms.

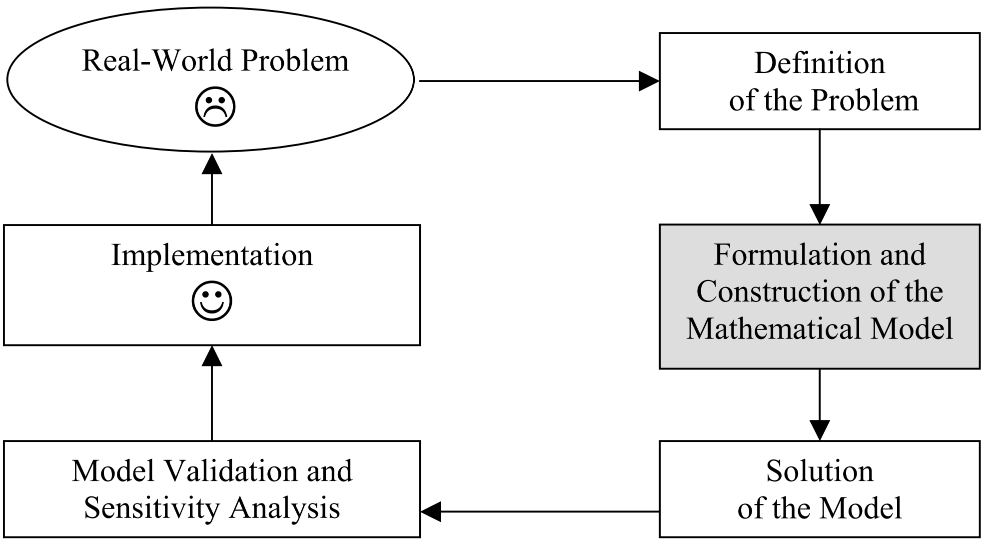


Fig. 1.1 Modeling Process

Nevertheless, the model is just simplified representation of the real world and therefore only important and relevant items or properties can be used in the model.

The first step is the **recognition** of the problem in reality and its **definition**. If we, for example, produce different products, the question is: how to set the production, i.e. how many of each product should be produced? In this step it is necessary to define goals that the management wants to achieve. In the example, it could be maximization of revenue or minimization of total cost. It is obvious that we have to know the details of production process and all the necessary information from marketing, finance, accounting, etc.:

1. what kinds of material are used and what is the available stock of them,
2. how many people can be involved in the production,
3. what is our budget,
4. how much material, energy, time and money is necessary for producing one unit of each product, how many people are needed,
5. unit prices of the final products (revenue maximization), unit cost of each product (total cost minimization),
6. the level of demand for each product.

The following step is the key issue of the modeling process – **formulation** and **construction** of a **mathematical model**. The problem defined above is being transformed into the mathematical form, i.e. into the world of numbers, symbols, parameters, variables, functions, equations, inequalities, etc. In this case, **variables** correspond to the amounts of products, **parameters** are the material stock, budget, unit prices, etc. There are several **constraints** (equations, inequalities) in the model, which do not allow us to exceed production possibilities and which assure that the produced amount is on the demanded level. The model includes the **objective function** for revenue (or total cost). All these terms are treated in detail in Chapter 2.

Construction of the model is doubtless the most difficult and the most important part of the whole process. Remember, the model is just the simplification of reality. The simpler the model, the easier its manipulation and solution will be. However, the simplest models could be far from reality and it has no sense to implement the results because of their low significance. The most complex model, e.g. with exact (nonlinear) relationships between variables, can be perfect representation of reality; it is hardly solvable. Whenever we start to construct a new model, it is necessary to remember an important rule:

> Finding a proper balance between the level of simplification of the model and the good representation of reality.

There are two kinds of models: **deterministic** and **probabilistic**. In deterministic models all aspects are known with certainty. In the above example, we can e.g. assume that the stock level, budget or used energy are exact numbers. Similarly, we exactly know how much of raw material will be used per each unit of a specific product. Deterministic models are not certainly perfect, but they may offer a reasonably good approximation of reality. Probabilistic models contain a specific level of uncertainty. Low level of uncertainty can be ignored and deterministic models can be used instead of the probabilistic models. In case of high level of uncertainty we should consider random variables instead of constants. In the illustrative example, we should assume that the demand for our products cannot be determined exactly. Probabilistic models, of course, require different (mostly more complex) approach to their solutions, the methods therefore differ from the methods used for deterministic models.

Solution to the proposed model is more or less a technical job. When we make a decision on the form of applied model, we should consider methods, techniques or principles used for the solution. Each area of management science gives typical examples of problems and suggests specific ways of solution. Knowledge of these typical examples facilitates the selection of the model form and the used methods. The typical goal of the most problems is to find an **optimal solution**, i.e. the best of all **feasible solutions** (solutions that satisfy all the constraints). In the example, the optimal solution is the one, which gives the highest revenue (the lowest total cost), while respecting all the constraints. This solution determines how many items of each product we should produce and what revenue (total cost) we can expect.

Expansion of computers brought new opportunities for solving wide range of real problems and nowadays it is hardly possible to get results without the use of computers and efficient software. Professional software offers the user-friendly environment with easy control, enables flexible entering of inputs and provides detailed outputs with possibility of experiments with the model. From this point of view very important step is the **model validation** and **sensitivity analysis**. Validation of the model is its comparison with the real system. The better a model represents the reality, the more valid it is. General method for testing the validity of a model is to compare its performance with some past data available for the actual system.

If we want to examine the impact of changes in inputs on changes in outputs, we have a convenient tool for this purpose – sensitivity analysis. In the example, there is specific stock of raw material. The level of stock is mostly the limitative factor for the production and hence for the attainable revenue. The interesting question is: what will happen to the proposed production (and revenue) if we avail of one additional unit of any material? In case all the material is used for our production, we can reasonably expect that, using one additional unit, we are able to produce more products and gain more money.

The final and very important step of the modeling process is the **implementation** of the proposed solution to the real system. This is actually the main goal of management and the original purpose of the whole process - not the model itself, but adjustment of reality according to the recommendations ensuing from the results of the modeling process. In case we do not use the results in the real production process, all our effort was absolutely vain. On the contrary, if we constructed the model in a wrong way and we did not validate it, the applied results could seriously harm the real system. In order to achieve the best management results, each step must be carefully considered and cannot be skipped.

1.3 Management Science Techniques

In the previous section it was claimed that the kind of problem indicates the form of model and used techniques. The following list [8] presents typical managerial problems where management science techniques could be applied:

- Inventory control.
- Facility design.
- Product-mix determination.

- Portfolio analysis.
- Scheduling and sequencing.
- Merger-growth analysis.
- Transportation planning.
- Design of information systems.
- Allocation of scarce resources.
- Investment decisions (new plants, etc.).
- Project management – planning and control.
- New product decisions.
- Sales force decisions.
- Market research decisions.
- Research and development decisions.
- Oil and gas exploration decisions.
- Pricing decisions.
- Competitive bidding decisions.
- Quality control decisions.
- Machine setup problems in production.
- Distribution decisions.
- Manpower planning and control decisions.
- Credit policy analysis.
- Research and development effectiveness.

For solving these and many other problems the following techniques can be used:

Linear Programming
It is one of the best-known tools of management science. This approach mostly defines the problem as the maximizing (minimizing) a linear function, respecting the set of linear constraints. The example mentioned in Section 1.2 is a typical task that can be successfully solved by the techniques of linear programming.

Integer Linear Programming
Further requirements for the values of variables are added to the original linear model (i.e. the model with the linear function and linear constraints). All or some of the variable values have to be an integer. The special type of those variables is binary variable (0-1 variable), value of which can be 0 or 1. In such case we speak about **Binary Integer Linear Programming**. If only some variables in the model are defined as integer (binary), we speak about **Mixed Integer Linear Programming**.

Goal Programming
When several competing objectives have to be considered simultaneously, more powerful tool is needed. Goal programming is a special technique for dealing with such cases, usually within the framework of linear programming.

Distribution Models
A distribution problem is a special type of linear programming problem. There are two main types of distribution problems: the **Transportation Problem** and the **Assignment Problem**. The transportation problem deals with shipments from a number of sources to a number of destinations, whereas the assignment problem deals with finding the best one-to-one match for each of a given number of possible "candidates" to a number of proposed "positions".

Nonlinear Programming
Models used in this area of management science are similar to the models of linear programming; however there is an important difference between them: nonlinear models contain nonlinear objective function and/or some nonlinear constraints. Methods used for solving tasks from this area of management science are, of course, rather different from the linear programming methods.

Network Models
Some problems can be described graphically as a network (the set of nodes and arcs). Typical situation is a transportation network: cities (nodes) are connected to each other by roads (arcs). If we evaluate the network (in this case we are interested e.g. in distances between all the cities), the task is often to find the minimal distance from one city to all other cities. Some types of networks may be evaluated by capacities instead of distances and then the problem of maximal flow can be solved. In many problems, solved with use of the network models, the most important value is unit cost and the goal is to find the minimal total cost.

Project Management
In many situations the managers are responsible for planning, scheduling and controlling projects that consist of many separate jobs or tasks performed by a variety of departments or individuals. An execution of each job takes specific time. There are two basic methods for solving those problems: CPM (*Critical Path Method*) and PERT (*Program Evaluation Review Technique*). Both methods require the network representation of the problem.

Inventory Models
Inventory control is one of the most popular techniques, which helps managers to determine when and how much to order. The main goal is usually to find a proper balance between the inventory holding cost and the cost of executing an order. There are many various inventory models because of many various real inventory systems. We consider two separated classes of models: deterministic and probabilistic. In deterministic models the rate of demand is constant through the time, whereas in probabilistic inventory models the demand fluctuates through the time and can be expressed only in probabilistic terms.

Waiting Line Models
This area of management science techniques deals with the situations where units (e.g. customers) need to be served by a number of channels (e.g. vendors). As the number of vendors is limited, some of the units have to wait for the service in a queue. Hence the alternative name for the waiting line models: **Queuing Models**. In real situations both the process of units' arrivals and the service times are random and the probabilistic approach is necessary. Simple waiting line models can be solved analytically (exact solution using derivative formulas), whereas for complex queuing systems the technique of simulation is required. The main managerial goal in waiting line models is the decision about the number of service lines (finding a proper balance between size of the queue and the total service cost).

Simulation
When managerial problems become more complex, they are often impossible (or non-effective because of spent time and cost) to be solved using standard techniques. For this purpose, simulation approach is advantageous and in many cases it is the only way how to manage the problem. Simulation is a computer experimentation with a simulation model aimed at describing and evaluating the real system's behavior - the computer program

simulates the real system. The typical situations for successful use of simulation are complex waiting line models and inventory models.

Decision Analysis
These techniques can be used to select optimal strategies out of several decision alternatives. Managerial problems and appropriate tools are divided, according to the kind of manager's information, into three classes: decisions under *certainty* (deterministic), decision under *risk* (probabilistic) and decisions under *uncertainty*. We consider special tools for this purpose: decision tables and trees.

Theory of Games
This area is an extension of decision analysis to the situations with two or more decision makers. Simultaneous decisions (selected strategies) of all managers initiate an action that affects all decision makers (players), i.e. their profit, cost, etc. In some conflicts, there is a possibility for two or more decision makers to cooperate, while competing with the others. In economic theory we can find a typical case of strategic game - oligopoly model.

Forecasting
Forecasting methods support the manager's prediction of future aspects of a business operation. Statistics and econometrics offer many techniques based on time-series and regression analysis. The main managerial goal is to project future trends following the previous behavior of the system. The well known are the methods of Moving Averages, Least Squares, Exponential Smoothing, etc. Since the statistical significance is very important for these models, the manager's experience with hypotheses validation and statistical tests is necessary.

Multicriteria Decision Making
In many managerial problems the decision maker needs to consider multiple criteria. If we find a solution improving one criterion, it mostly worsens some of other criteria. It is usually impossible to optimize simultaneously all the criteria. The reasonable opportunity for the management is to find a suitable compromise. If the count of alternatives is limited, we use methods of alternatives evaluation. Some problems are described with the set of constraints and the set of objective functions. In this case the solution is provided by multiobjective programming techniques. The special category of this management science technique is goal programming.

Markov Analysis
This technique can be used to describe the behavior of a system in a dynamic situation (evolution of the system throughout the time). If - at a given time point - the system is in one of possible states, at following time point the system can remain in current state or can move into any other state. Remaining in current state or movement to another state are set by transition probabilities. The manager can be interested in the probability with which the system will be in the specific state at the specific time. Markov analysis is a very powerful tool of management science with many real applications.

Dynamic Programming
Management must frequently consider a sequence of decisions where each decision significantly affects future decisions. Dynamic programming helps managers to solve certain types of such sequential decision problems. There is no single model for solving dynamic programming problems and the problems are therefore classified into many groups. One

possible classification considers deterministic and probabilistic models. Models often use network representation of the sequential problems. Markov analysis can be considered as a probabilistic model of dynamic programming.

The survey presented by Anderson, Sweeney, and Williams [1] indicates that the most frequently used techniques are statistical methods, computer simulation, PERT/CPM, linear programming and queuing theory. Frequency of using all mentioned techniques, of course, depends on the specialization of the firm and its size. With respect to the limited space in this textbook and its purpose, we cannot go through all the management science techniques in detail and we will therefore point out the key methods and applications.

1.4 Glossary

Binary Integer Linear Programming – bivalentní programování
Special case of linear programming in which all the decision variables are binary.

Constraints – omezení
Restrictions on the problem solution arising from limited resources, policy requirements, etc.

Decision Making – rozhodování
A process of choosing between two or more available alternative courses of action for the purpose of attaining a specific goal.

Deterministic Model – deterministický model
A model in which the functional relationships and parameters are known with certainty.

Feasible Solution – přípustné řešení
A solution that respects all the constraints.

Goal Programming – cílové programování
A problem approach when several objective functions are considered simultaneously. The objective is to minimize the undesirable deviations from the goals.

Integer Programming – celočíselné programování
A programming approach that assumes the indivisibility of all the decision variables (e.g. products, people, etc.). The values of the variables must be integers.

Linear Programming – lineární programování
A mathematical procedure for optimizing the linear objective function, respecting the set of linear constraints.

Management Science
see Operations Research.

Mathematical Model – matematický model
A system of symbols and expressions aimed at representing a real situation.

Maximization – maximalizace
Optimization of objective function, which looks for the highest objective value (e.g. profit, revenue).

Minimization – minimalizace
Optimization of objective function, which looks for the lowest objective value (e.g. cost, loss).

Mixed Integer Linear Programming – smíšeně celočíselné programování
Special case of integer linear programming in which some but not all of the decision variables are integer.

Model – model
An abstraction of reality.

Nonlinear Programming – nelineární programování
A problem approach used when the objective function and/or one or more constraints are nonlinear.

Objective Function – účelová funkce
A mathematical function expressed in terms of decision variables, which is to be optimized (maximized or minimized).

Operations Research – operační výzkum
The application of the scientific method to the study and analysis of problems involving large and complex systems, organizations or activities.

Optimal (Optimum) Solution – optimální řešení
A feasible solution that maximizes or minimizes the objective function. The best of all feasible solutions.

Probabilistic Model – stochastický model
A model that incorporates uncertainty in its functional relations and uncontrollable variables.

Sensitivity Analysis – analýza citlivosti
Measuring the effect of a change in one input parameter on a proposed solution.

2. Linear Programming

Linear Programming is a tool of management science for solving optimization problems. The word “linear” indicates that all mathematical relationships in a model are linear. The typical model consists of the set of linear equations and/or inequalities (called **constraints**) and the linear **objective function** (which is to be **maximized** or **minimized**). The **nonnegativity constraints** (variables are zero or positive) are mostly involved in the model. The manager’s goal is to find the **optimal solution** with the best value of the objective function.

2.1 Formulation of the Mathematical Model

Example 2.1

Pinocchio, Inc. manufactures 2 types of wooden toys: trucks and trains. The price of a piece of truck is 550 CZK, of a piece of train 700 CZK. The wood cost for the truck is 50 CZK, whereas for the train 70 CZK. The truck requires 1 hour of carpentry labor and 1 hour of finishing labor (assembling and painting). The train requires 2 hours of carpentry labor and 1 hour of finishing labor. Worth of carpentry labor is 30 CZK per hour, worth of finishing labor is 20 CZK per hour. Each month, Pinocchio has 5000 available hours of carpentry labor and 3000 hours of finishing labor. Demand for trains is unlimited, but at most 2000 trucks are, at an average, bought each month. The Pinocchio’s management wants to maximize monthly profit (total revenue - total cost).

Formulation

1. **Decision variables**

The variables should completely describe the decisions to be made by the management. The manager must decide how many trucks and how many trains should be manufactured each month in order to maximize the profit. In this case the decision variables are:

x_1 = number of trucks produced each month,
x_2 = number of trains produced each month.

2. **Objective function**

This function represents the management’s criterion that is to be maximized or minimized. In the Pinocchio’s situation the management intends to maximize total monthly profit as the difference between total monthly revenue and total monthly cost. Both revenue and cost can be expressed as the function of decision variables x_1 and x_2.

1. Total revenue (TR) = revenue from sold trucks + revenues from sold trains.

Since the price of one truck is 550 CZK and the manufacturer produces x_1 of trucks, the revenue ensuing from all the realized trucks is $550x_1$. Similarly, the revenue from sold trains is $700x_2$.

The total monthly revenue ensuing from the production is then expressed as:

$TR = 550x_1 + 700x_2$.

2. Monthly wood cost (*WC*) = *WC* of produced trucks + *WC* of produced trains.

If we know the wood cost of production of one truck (50 CZK) and the total number of produced trucks is x_1, the monthly wood cost of all produced trucks is $50x_1$. Similarly, the monthly wood cost of all trains is $70x_2$.

The total monthly wood cost is then:

$$WC = 50x_1 + 70x_2 .$$

3. Carpentry labor cost (*CLC*) = *CLC* of produced trucks + *CLC* of produced trains.

If one truck requires 1 hour of carpentry labor and cost of 1 hour of this labor is 30 CZK, the unit cost is 30 CZK. The monthly cost of carpentry labor used for all produced trucks is $30x_1$. Since one train requires 2 hours, the monthly cost of carpentry labor used for trains is $60x_2$.

The total monthly cost of carpentry labor is:

$$CLC = 30x_1 + 60x_2 .$$

4. Finishing labor cost (*FLC*) = *FLC* of produced trucks + *FLC* of produced trains.

Both a piece of truck and a piece of train require 1 hour of finishing labor. Cost of this labor is 20 CZK per hour. Hence the total monthly cost of finishing labor:

$$FLC = 20x_1 + 20x_2 .$$

5. The total monthly cost can be expressed as:

$$TC = WC + CLC + FLC = (50x_1 + 70x_2) + (30x_1 + 60x_2) + (20x_1 + 20x_2),$$
$$TC = 100x_1 + 150x_2 .$$

6. The total profit, as the objective, is the difference between the total monthly revenue and the total monthly cost:

$$TP = TR - TC = (550x_1 + 700x_2) - (100x_1 + 150x_2) = 450x_1 + 550x_2 .$$

In the linear programming model the objective function is expressed as:

$$\boxed{\text{Maximize } z = 450x_1 + 550x_2 .}$$

Numbers 450 and 550 in the function are called *objective function coefficients*.

3. Constraints

If there are no restrictions, objective function (profit) can grow to infinity. However, there are three restrictions (called constraints) for the toys production:

1. Each month Pinocchio, Inc. has only 5000 available hours of carpentry labor.

2. Each month no more than 3000 hours of finishing labor may be used.
3. Because of limited demand, at most 2000 trucks should be produced each month.

We express these three constraints in the mathematical way:

1. One truck requires 1 hour of carpentry labor. If the manufacturer produces monthly x_1 of trucks, x_1 hours of labor are used. Considering that one train requires 2 hours and the production quantity equals x_2, the monthly use of carpentry labor is $2x_2$ hours.

The total use of carpentry labor for both products can be expressed as $x_1 + 2x_2$. This is actual use of labor (in hours) that cannot be greater than available number of hours (5000). With this in mind, the constraint can be expressed as:

$$\boxed{x_1 + 2x_2 \le 5000\,.}$$

2. The construction of the second constraint concerning finishing labor is similar to the previous one:

$$\boxed{x_1 + x_2 \le 3000\,.}$$

3. The last constraint is very easy to be built. The number of produced trucks x_1 must be less than or equal to 2000:

$$\boxed{x_1 \le 2000\,.}$$

The coefficients of the decision variables in the constraints are called *technological coefficients*, numbers 5000, 3000 and 2000 are called *right-hand side values*.

4. Nonnegativity constraints

There are reasonable sign restrictions associated with both the decision variables: since the values of variables represent numbers of produced toys we should expect them not to be negative:

$$\boxed{x_1, x_2 \ge 0\,.}$$

The mathematical model can be summarized in standard form as follows:

$$\boxed{\begin{array}{l} \text{Maximize } z = 450x_1 + 550x_2\,, \\ \text{subject to} \\ x_1 + 2x_2 \le 5000\,, \\ x_1 + x_2 \le 3000\,, \\ x_1 \le 2000\,, \\ x_1, x_2 \ge 0\,. \end{array}}$$

2.2 Graphical Solution of Linear Programming Problems

If a linear programming problem contains only two decision variables, it is possible to solve it graphically. The following steps lead to the solution:

Step 1 - Graphing a feasible area

Each constraint (including nonnegativity constraints) can be very easily drawn; since all the constraints must be satisfied simultaneously, the combination of drawn constraints determines the *feasible area*. Each point from the feasible area corresponds to the *feasible solution* of the problem.

In the Pinocchio's problem there are 3 constraints and 2 nonnegativity constraints. Nonnnegativity constraints are easy to graph, because all combinations of x_1 and x_2 must lie in the first quadrant of the x_1 - x_2 plane (Figure 2.1).

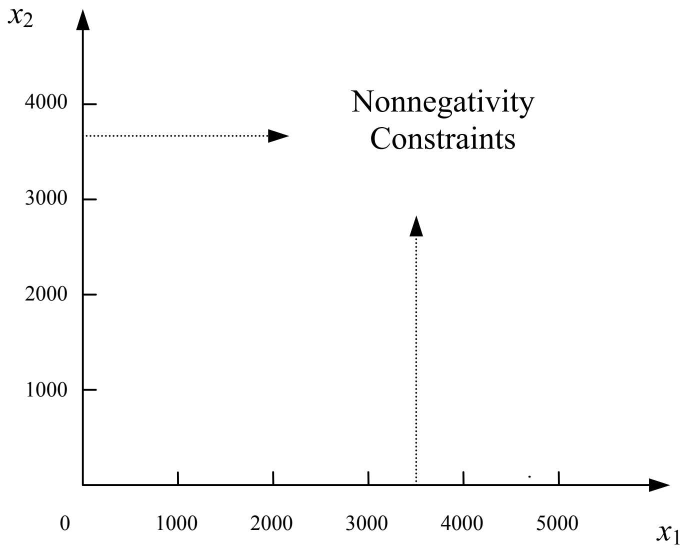


Fig. 2.1 Nonnegativity Constraints

For graphing a linear inequality (carpentry labor constraint)

$$x_1 + 2x_2 \leq 5000$$

it is necessary to draw the borderline, which corresponds to the linear equation

$$x_1 + 2x_2 = 5000.$$

In order to get the borderline, we must find coordinates of its two points. The easiest way is to determine the borderline's intersection points with axes x_1 and x_2. If x_1 is set to zero (i.e. no trucks are produced), then $x_2 = 5000/2 = 2500$ (2500 trains can be produced). This situation refers to the point A (0, 2500) in Figure 2.2.

Similarly, if x_2 is set to zero (no trains are produced), then $x_1 = 5000$ (5000 trucks can be produced). It corresponds to the point B (5000, 0) in Figure 2.2.

Joining points A and B by a straight line we get a graphical representation of the equation $x_1 + 2x_2 = 5000$. Now we have to decide which part of first quadrant (divided by the line AB) corresponds to the inequality $x_1 + 2x_2 \leq 5000$.

The general method for this purpose is to select an arbitrary point (of course not from the line) and check whether it is feasible. If the point is feasible, then the entire area is feasible as well.

In our situation the easiest way is to take the origin (0, 0). After substitution of these coordinates into the constraint, we get $0 \leq 5000$ (the constraint is satisfied). Hence, the area including the origin is feasible (shaded in Figure 2.2).

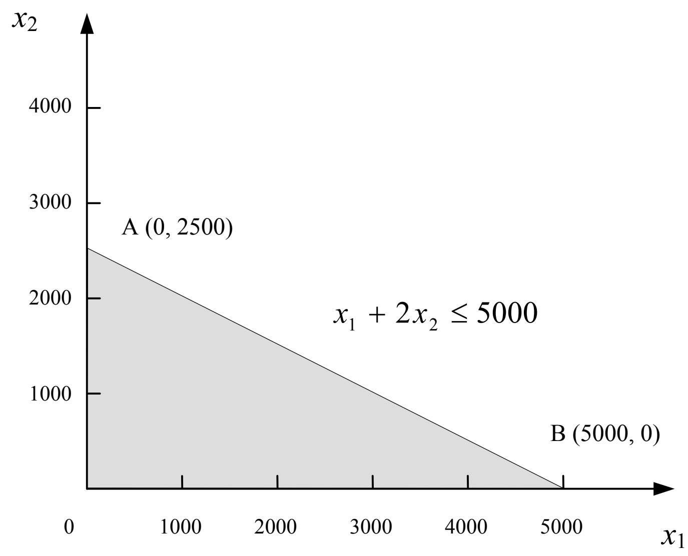


Fig. 2.2 Carpentry Labor

Note: Graphing the feasible area, we excluded negative values of x_1 and x_2 because of nonnegativity constraints (Figure 2.1).

The second inequality $x_1 + x_2 \leq 3000$ (finishing labor constraint) is drawn in Figure 2.3, the last constraint $x_1 \leq 2000$ can be seen in Figure 2.4. The equation $x_1 = 2000$ is represented by a line parallel to the axis x_2, the feasible area is on the left side of the line.

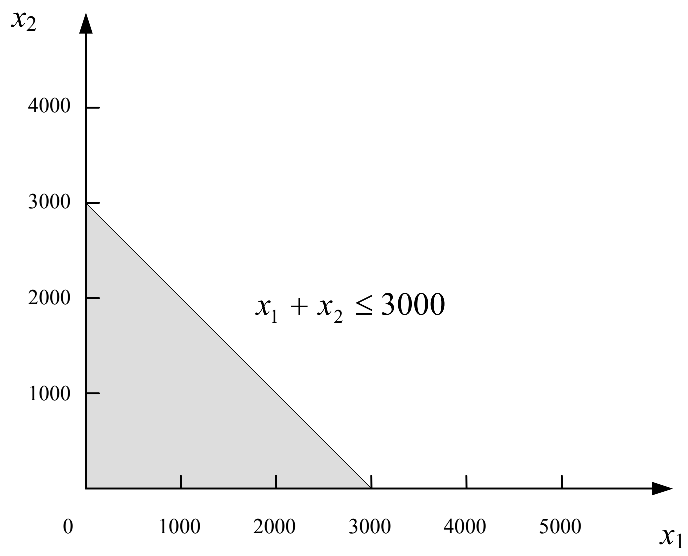


Fig. 2.3 Finishing Labor

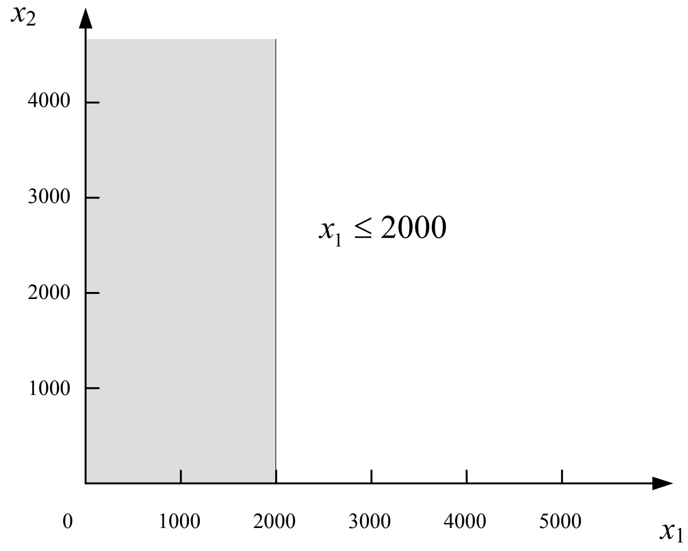


Fig. 2.4 Limited Demand

Step 2 - Combining the constraints

Once all the constraints have been individually drawn, they must be put together in one graph. Practically, it is possible to build this final graph from the beginning by sequential adding of the individual feasible areas.

Figure 2.5 shows the final graph of the feasible area in the example. Any solution in this area is feasible with respect to all of the constraints.

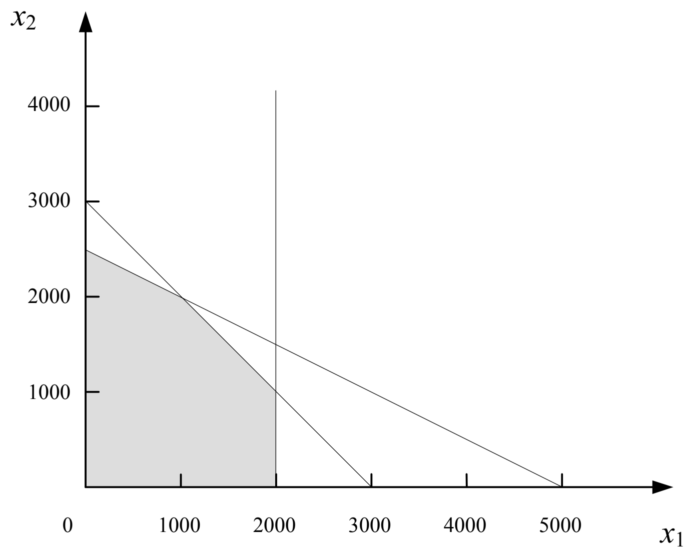


Fig. 2.5 Feasible Area

Step 3 – Graphing an objective function

In the example, the management's objective is expressed as the profit function $z = 450x_1 + 550x_2$. This function cannot be obviously presented as a single line because of infinite number of equations. The specific equation depends on the value of z.

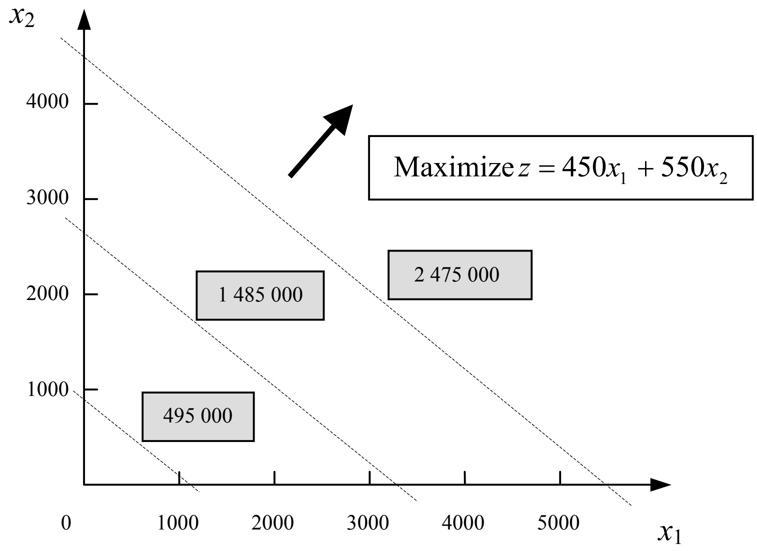


Fig. 2.6 Isoprofit Lines

Figure 2.6 shows three of those equations with values of 495 000, 1 485 000 and 2 475 000. All the lines are parallel to each other and can be built similarly as the borderlines of constraints in Step 1. All points on specific line give the same profit and therefore the lines are usually called *isoprofit lines*. The arrow indicates the growth of the profit.

Step 4 – Finding the optimal solution

Combining the feasible area and the isoprofit lines, we can find the isoprofit line which is as far as possible from the origin, but still touches the feasible area.

In Figure 2.7 the dashed isoprofit line corresponds to the maximal profit z = 1 550 000 CZK. This line intersects one corner point of feasible area, called an *optimal solution*. It is the point X (1000, 2000).

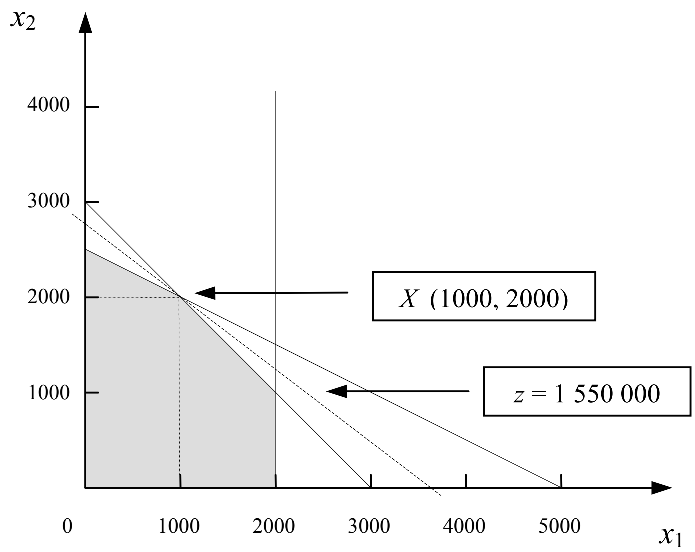


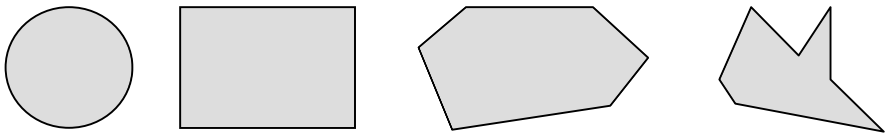
Fig. 2.7 Optimal Solution

There are two possible methods how to find the optimal solution using graphical representation of the model:

1. The first method has been just described. Finding a slope and the growth direction of the isoprofit line, we can determine the “best” point of the feasible area. This point is the optimal solution and the isoprofit line corresponds to the optimal value of the objective function. In this part of text it is reasonable to note that the feasible area of all linear programming models is a convex set.

   A set of points S is a **convex set** if the line segment joining any pair of points in S is wholly contained in S.

   Figure 2.8 illustrates the difference between convex and non-convex sets. Whereas the first three cases (a), (b) and (c) are convex sets, the set (d) is non-convex.

   (a) (b) (c) (d)

   Fig. 2.8 Convex and Non-Convex Sets

   The set (a) slightly differs from the sets (b) and (c), because of nonlinear border. If the border of the convex set consists of linear segments only, as it is shown in cases (b) and (c), the set is called a *convex polyhedron* and it represents the typical feasible area of linear programming models. It is the useful attribute used in linear programming methods.

   The important question is how to calculate the coordinates of the optimal solution $X(1000, 2000)$. As it is obvious from Figure 2.6, this point is the intersection of two constraints' borderlines. It is therefore quite easy to determine the coordinates by solving the set of two linear equations. In our case the borderlines equations are:

   $$x_1 + 2x_2 = 5000 ,$$
   $$x_1 + \ \ x_2 = 3000 .$$

   The solution is really simple: $x_1 = 1000$ and $x_2 = 2000$. These values are the coordinates of the point X. Introducing them into objective function $z = 450x_1 + 550x_2$, we get the objective value $z = 1\,550\,000$.

2. Before starting to describe the second method used to find the optimal solution in the graph, it is necessary to define a *corner point*. Referring to the term of the convex polyhedron it is possible to define corner point simply as follows:

   A point P in convex polyhedron S is a **corner point** if it does not lie on any line joining any pair of other (than P) points in S.

The *basic linear programming theorem* is the keystone of the second method:

> The optimal feasible solution, if it exists, will occur at one or more of the corner points.

Following this theorem, all we need is to identify the corner points of the feasible area and calculate the objective values for all these points. The optimal solution is the corner point with the best objective value.

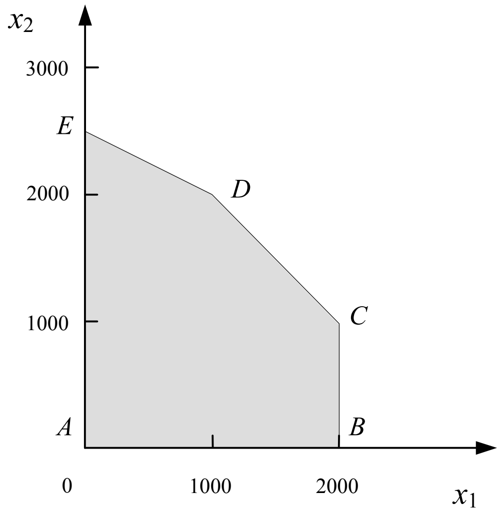


Fig. 2.9 Corner Points

In the example, the feasible area contains 5 corner points (Figure 2.9), the coordinates of which can be found in Table 2.1. All the coordinates have been computed in the way described above: after identification of the intersected lines, the set of equations is solved and the solution is given by the coordinates x_1 and x_2. For each solution the objective value z is then computed by introducing the coordinates into the objective function. It is evident from the table that the optimal solution corresponds to the corner point D with the maximal objective value z = 1 550 000 CZK.

Corner point	x_1	x_2	z
A	0	0	0
B	2000	0	900 000
C	2000	1000	1 450 000
D	1000	2000	1 550 000
E	0	2500	1 375 000

Tab. 2.1 Corner Points

Simplex method is a general method used for solving linear programming problems. It is an iterative algorithm for efficient searching for the optimal solution. The method uses the Gauss-Jordan method of solving simultaneous equations. In addition, the method is based on the basic linear programming theorem (the search can concentrate only on the corner points of the feasible area). The run of algorithm starts in one of the corner points (usually in the origin) and moves to adjacent corner that improves the value of the objective function. The process of movement continues until no further improvement is possible. The simplex method and its variations have been programmed and therefore even large linear programming problems can be easy solved. More about the simplex algorithm see [1] or [8].

2.3 Interpreting the Optimal Solution

The correct and precise interpretation of the results is as important as the model itself. After obtaining the solution, it is necessary to return to the beginning of the modeling process and compare the variables to their real representations. In the example, x_1 is the number of trucks produced each month, and x_2 is the number of trains produced each month. As the optimal solution we have got $x_1 = 1000$ and $x_2 = 2000$, therefore it is optimal to produce 1000 trucks and 2000 trains each month. If the management follows this production structure, Pinocchio, Inc. makes profit 1 550 000 CZK each month (see the definition of the objective function in the model).

If the production differed from the computed solution but still respected all restrictions of the model, the profit would be lower. The management cannot make higher profit than 1 550 000 CZK without breaking at least one restriction. The solution would be infeasible.

Interpreting the results we should be interested in actual use of available resources (hours of labor) and in achievement of limited demand. If we introduce the optimal values of x_1 and x_2 into all the constraints, we will see whether they are satisfied exactly as equations or as inequalities. As for carpentry labor, the left-hand side value of the constraint is 1000 + 2(2000) = 5000, which exactly equals the right-hand side value 5000 (the constraint is *binding*). Since the right-hand side corresponds to the available hours of labor, the equality means that no hour remains, or all the available hours are used for the optimal production.

The situation of finishing labor is similar. The left-hand side value of the constraint is 1000 + 2000 = 3000, right-hand side value equals 3000 as well; thus all the available hours of finishing work are used. The last constraint concerns the limitation of demand (the number of produced trucks should be less than or equal to 2000). Since it is optimal to produce 1000 trucks (the constraint is *nonbinding*), this is not the limitation that affects the production of Pinocchio, Inc. The company could expand the production of trucks when it managed to get more hours of labor or to improve the technology (i.e. to decrease the unit consumption of labor).

Note: For the inequality of type "≤" we define a *slack variable*. This variable evaluates a difference between the right-hand side and the left-hand side of the constraint. In the example, the slack variables in both labor constraints are zero, as both sides of the constraints equal to each other. The slack variable in demand constraint is 1000, since the right-hand side value is 2000 and left-hand side value is 1000. For the inequality of type "≥" we define a *surplus variable* evaluating the difference between the left-hand side and the right-hand side of the constraint. If we should produce, for example, at least 600 trucks because of demand requirements ($x_1 \geq 600$) and we actually produce 1000 trucks, the surplus variable is 400.

Calculation of the impact of the changes in a real system on the optimal solution is the principle of a *post-optimal (sensitivity) analysis*. We projected possible changes in the stock of resources (available hours of labor), in the technology, but it is reasonable to analyze how the changes in the product prices touch the profit.

In Figure 2.10 we compare the original objective function $z = 450x_1 + 550x_2$ with the function $z = 750x_1 + 550x_2$ (the price of the truck changes from 550 CZK to 850 CZK). We get a new optimal solution (2000, 1000) with the profit z = 2 050 000 CZK.

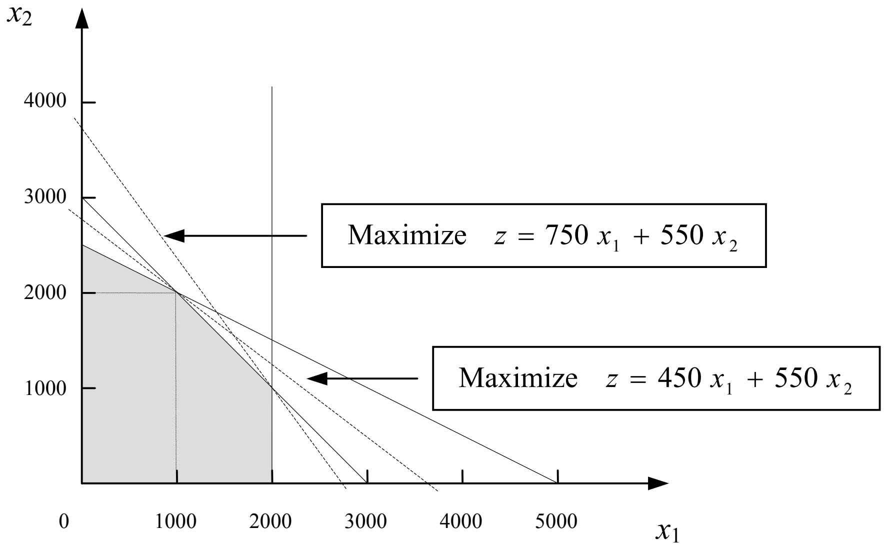


Fig. 2.10 Change in the Product Price

Sensitivity analysis determines, mostly for coefficients of objective function and right-hand side values, the range of their possible changes that do not have the cardinal influence on the optimal solution. If we changed the truck's price from 550 CZK to 649 CZK (i.e. the objective function coefficient was changed from 450 to 549), the optimal solution would be the original corner point (1000, 2000). The only change would be higher profit, while the production structure would be identical. If the price increased to 651 CZK, the optimal solution (2000, 1000) would be obtained, and following this recommendation the company should change the production structure from the base. Note that price 650 CZK gives to the company the possibility to produce either 1000 trucks (and 2000 trains) or 2000 trucks (and 1000 trains). This special situation will be discussed later.

The change in coefficients of objective function influences only its slope, whereas the feasible area remains fixed. The slope of the function determines the selection of the best corner point. When any right-hand side value is changed the feasible area is changed as well.

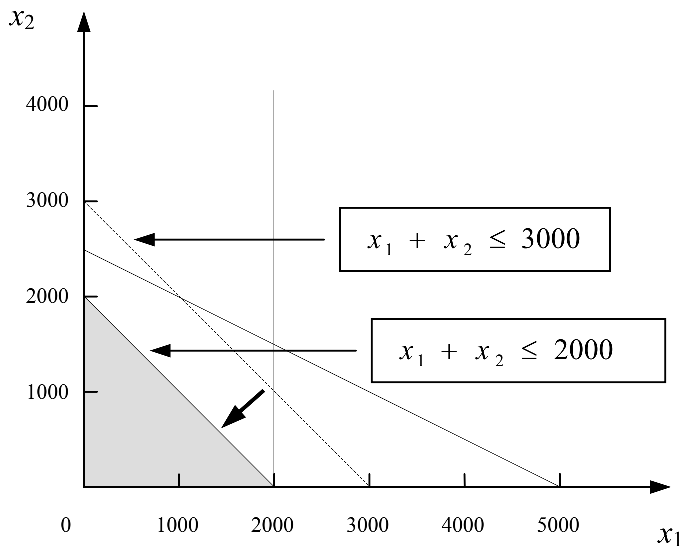


Fig. 2.11 Change in Available Finishing Labor

The change causes the movement of the constraint's borderline (the new borderline is parallel with the original). The smaller the change in the right-hand side value, the smaller the change in the feasible area will be. Big changes can make the cardinal change in the feasible area as Figure 2.11 shows (decrease in available finishing labor from 3000 to 2000 hours). The original number of the corner points 5 decreases to 3. The new optimal solution, in this situation, is represented by the corner point (0, 2000).

Since the optimal corner point is out of the carpentry labor constraint, there are some unused hours of this labor, i.e. the slack variable in this inequality becomes nonzero now. The exact unused amount can be simply determined by introducing the coordinates (0, 2000) into the left-hand side of the constraint. As 0 + 2(2000) = 4000, the number of unused hours is 5000 - 4000 = 1000.

Note: All the changes should be studied separately, i.e. only one parameter is being changed (the others are fixed) and we analyze the impact on the optimal solution and the profit. The integral parts of the sensitivity analysis are, together with the *ranges of objective function coefficients* and the *ranges of right-hand sides*, the *reduced costs* and the *shadow prices*. More about these terms see [1] or [8].

2.4 Special Cases of Linear Programming Models

In the original version of the production problem in Pinocchio, Inc. the only **one optimal solution** has been obtained. It is a typical case of the most linear programming problems. Illustrative Figure 2.12 shows the feasible area together with the objective function z (the arrow indicates the direction of improvement of the objective value). The corner point A is the unique optimal solution.

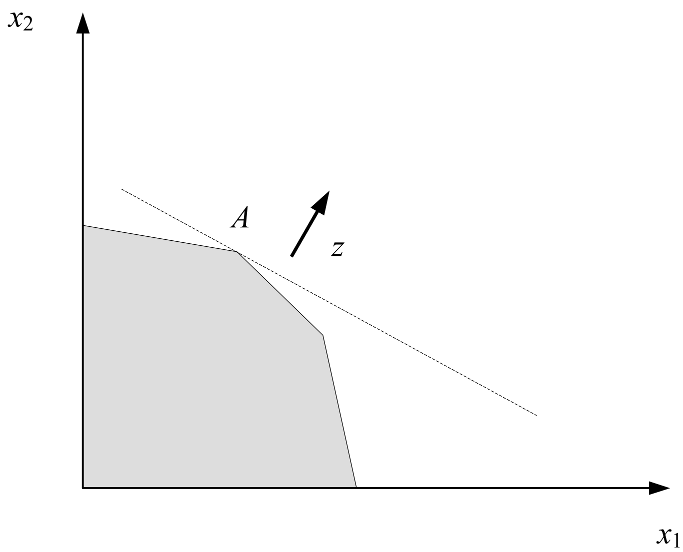


Fig. 2.12 Unique Optimal Solution

In the previous section, when describing the post-optimal analysis of the objective function coefficients, one special situation was mentioned. When the price of the truck is 650 CZK (instead of 550 CZK), the management of Pinocchio, Inc. has two basic possibilities how to optimize the profit. Producing 1000 trucks (and 2000 trains) or 2000 trucks (and 1000 trains) would bring the identical optimal profit 1 650 000 CZK.

A linear programming problem with two ore more optimal solutions is said to have **alternative (or multiple) optimal solutions**.

In graphical representation of the model this situation appears when the objective function line is parallel to one constraint's borderline as we can see in Figure 2.13. Since the objective function represents the isoprofit line (in case of maximization) or the isocost line (in case of minimization), all the solutions on the edge of feasible area have the identical (optimal) objective value. There are two optimal corner points (*B* and *C*) and the infinite number of optimal points on the line segment *BC*.

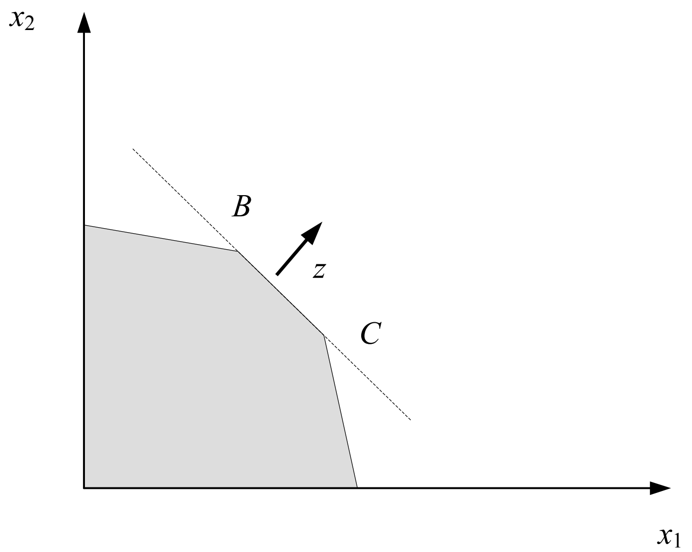


Fig. 2.13 Multiple Optimal Solutions

Figure 2.14 shows an interesting alternative of just described case. The difference is evident – the feasible area is unbounded and there is only one optimal corner point *D*; the other optimal points lie on the borderline running to infinity.

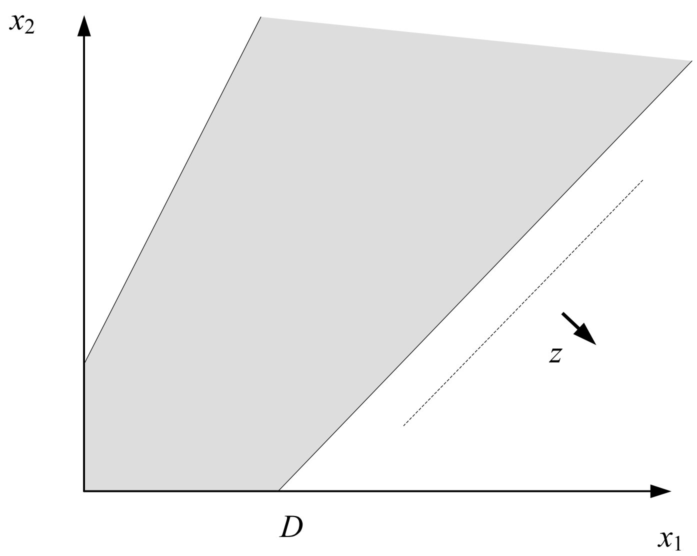


Fig. 2.14 Multiple Optimal Solutions – Unbounded Feasible Area

In case that the feasible solution area is unbounded and the objective value is being improved in the direction of unboundedness, **no optimal solution** can be found, in other words, the **optimal solution is infinite** (see Figure 2.15). Since such case hardly occurs in practice, this result usually means an error in the formulation.

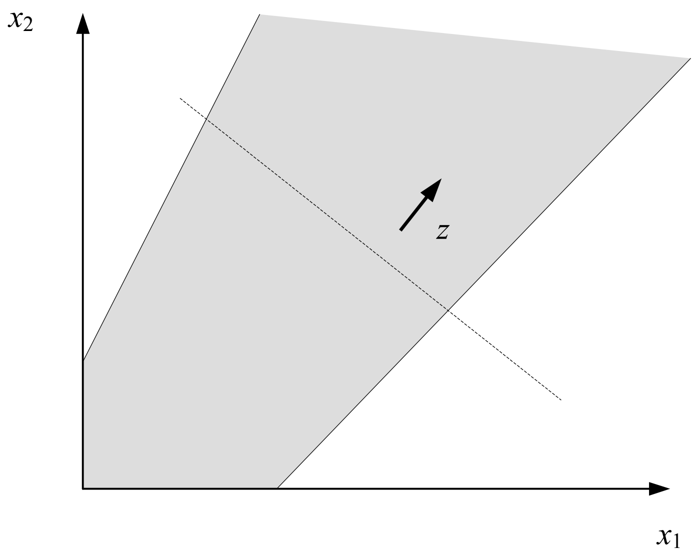


Fig. 2.15 No Optimal Solution

Infeasibility occurs whenever there is no solution that satisfies all the constraints. In case of two constraints, graphed in Figure 2.16, there is no common feasible area (no feasible solution). This situation often happens in practice, especially in case the manager intends to be too precise in formulation of the model. To eliminate the infeasibility it is necessary to simplify the model e.g. by excluding some vain constraints.

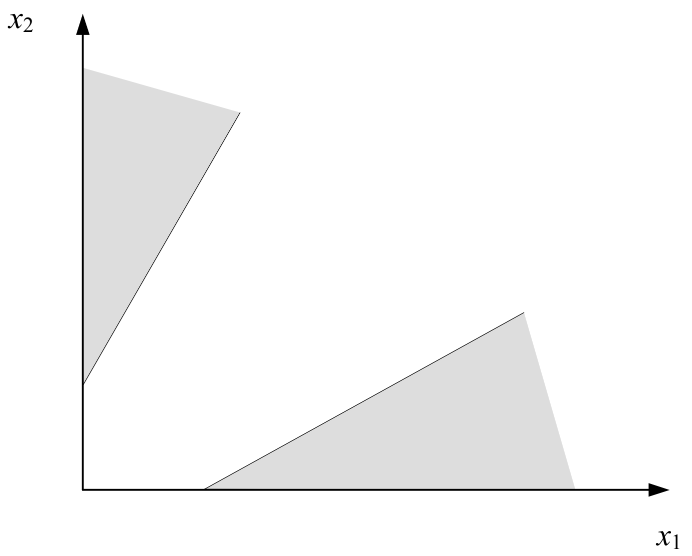


Fig. 2.16 No Feasible Solution

Although the two-variable model is not frequent in practice and is rather used for methodological purposes, all the ideas can be generalized and transformed for multidimensional problems.

2.5 Applications

Linear programming models are often and successfully used in many practical situations. Simplification of the complex reality is the key issue that managers must consider during all the modeling process. Formulation of linear programming models is an art that can be mastered just with practice and experience. Although each problem has unique features, many problems have some common features and fall into the same category of problems.

We offer the following general guideline for model formulation:

1. **Understand the problem thoroughly.** Read the description of the problem carefully and identify all necessary items that must be included in a model.

2. **Write a verbal statement of the objective function and each constraint.** Since all the statements will be later transformed into the mathematical model, the verbal description should be as precise as possible. The objective function (criterion) statement can be written as, for example, "maximize monthly profit (in CZK)". The constraint (restriction) might be specified as "actually used finishing labor (in hours) must not exceed available finishing labor (in hours)".

3. **Define the decision variables.** The variables and their values represent the manager's decisions. Remember these variables will be used in the mathematical model (in the objective function and in all the constraints); therefore their definition must be clear and unique. Do not forget to consider nonnegativity constraints and/or the integer (or bivalent) type of the variables. Dimension and appropriate unit for each variable must be defined as well.

4. **Write the objective function in terms of the decision variables.** Transform the verbal statement of the objective (Step 2) into the mathematical statement. In linear programming models the mathematical representation of the objective must be linear function of the decision variables.

5. **Write the constraints in terms of the decision variables.** Transform the verbal statement of each constraint (Step 2) into the mathematical statement in the form of the equation or the inequality. The left-hand side of each constraint must be a linear function of the decision variables and the right-hand side must be a nonnegative constant.

This brief guideline can be very helpful especially for beginners. After formulating the model the manager makes a decision about the solution of the problem, usually using computer and suitable software. It seems to be very easy to transform the mathematical model into a computer input and it is more or less a technical job. Used software usually provides the optimal solution (values of all decision variables), the optimal objective value, the values of the slack/surplus variables and parameters concerning the sensitivity analysis of the optimal solution.

As the formulation of the model is the most difficult issue of the whole modeling process, it might be useful to present some examples of real problems with a projection of their mathematical models.

2.5.1 Production Process Models

There are many variations of this category of problems. In Section 2.1 one of them (Pinocchio, Inc.) has been formulated. Typically the management has to make a decision which products to produce and how many units of each product should be produced. The production is described by technological (input-output) coefficients; the stock of used resources is mostly limited. Additional requirements corresponding, for example, to the level of demand could be considered in the problem definition. The most frequent criteria of those models are profit (or revenue) maximization and cost minimization. Since the issue of Pinocchio, Inc. has been analyzed in this Chapter, we will not be discussing these models any more.

2.5.2 Blending Problems

Real situations in which some inputs should be mixed to produce different outputs are called the blending problems. The following list [9] gives the examples of such a linear programming problem:

1. Blending various chemicals to produce other chemicals.
2. Blending various types of metal alloys to produce various types of steel.
3. Mixing various types of paper to produce recycled paper of varying quality.
4. Blending various livestock feeds in attempt to produce a minimum-cost feed mixture for cattle.
5. Blending various types of crude oils to produce different types of gasoline.
6. Mixing various ingredients (meat, salt, water, pepper, chilli, etc.) to produce different smoke-meat products.

Mathematical model of the blending problem is quite similar to the models of production process; the difference consists in the definition of the decision variables. Whereas in the production process the variables correspond almost with outputs (number of produced products), in the blending problem the variables refer to the inputs (amount of ingredients used in the final blend).

2.5.3 Marketing Research

Example 2.2

MarketQuest, Inc. specializes in evaluating consumer's reaction to new products and services. A company, introducing new type of washing powder, asked the MarketQuest, Inc. to prepare a campaign with door-to-door personal interviews about households' opinion. Households both with children and without children should be interviewed; both daytime and evening interviews should be conducted. There is a plan to conduct 1000 interviews with the following restrictions:

a) At least 300 households with children should be interviewed.
b) At least 400 households without children should be interviewed.
c) The total number of evening interviews should be greater than or equal to the number of daytime interviews.
d) At least 35% of the interviews for households with children should be conducted during evening.
e) At least 65% of the interviews for households without children should be conducted during evening.

The cost varies with the type of interview. Interview for household with children is more complex and longer; evening interviewers are paid more than daytime interviewers. The estimation of the cost is as follows:

	Daytime interview	Evening interview
Households with children	50 CZK	60 CZK
Households without children	40 CZK	50 CZK

Tab. 2.2 Interview Cost

The project manager of MarketQuest, Inc. should suggest the schedule of the interviews that will satisfy the restrictions with the minimal total cost.

Formulation

1. Define the decision variables

x_1 = the number of daytime interviews for households with children,
x_2 = the number of evening interviews for households with children,
x_3 = the number of daytime interviews for households without children,
x_4 = the number of evening interviews for households without children.

2. The objective function to be minimized is the total interview cost:

Minimize $z = 50x_1 + 60x_2 + 40x_3 + 50x_4$.

3. The constraints

a) At least 300 households with children should be interviewed:

$$x_1 + x_2 \geq 300 .$$

b) At least 400 households without children should be interviewed:

$$x_3 + x_4 \geq 400 .$$

c) The total number of evening interviews should be grater than or equal to the number of daytime interviews:

$$x_2 + x_4 \geq x_1 + x_3 ,$$

or re-written in standard format of linear programming model (with constants in right-hand side):

$$-x_1 + x_2 - x_3 + x_4 \geq 0 .$$

d) At least 35% of the interviews for households with children should be conducted during evening:

$$x_2 \geq 0.35\,(x_1 + x_2) , \quad \text{or} \quad -0.35x_1 + 0.65x_2 \geq 0 .$$

e) At least 65% of the interviews for households without children should be conducted during evening:

$$x_4 \geq 0.65\,(x_3 + x_4)\,, \quad \text{or} \quad -0.65x_3 + 0.35x_4 \geq 0\,.$$

f) Totally 1000 interviews should be conducted:

$$x_1 + x_2 + x_3 + x_4 = 1000\,.$$

4. The nonnegativity constraints and the integer type of all variables

$x_1, x_2, x_3, x_4 \geq 0$,

x_1, x_2, x_3, x_4 are integers.

Optimal Solution

$x_1 = 195$,
$x_2 = 105$,
$x_3 = 245$,
$x_4 = 455$.

$z = 48\,600$ CZK.

2.5.4 Portfolio Selection Problem

In this type of problem the financial manager must select specific investments (shares, bonds, etc.) among a limited number or various investment alternatives. These situations are typical for mutual funds, credit unions, banks and insurance companies. The objective function of managers is usually the maximization of expected return or the minimization of risk. Portfolio selection problem can be formulated in many model variations (usually as the model of nonlinear programming), however we focus on the linear programming model.

Example 2.3

Drink Invest, Inc. is a company interested in investing money in stocks of companies producing drinks. The management of the company speculates on investing 2 000 000 CZK in four available shares. To prevent a high risk index there is a necessity of investing some amount of money in government bonds. From a long-term survey of the financial market following annual rates of return and risk indices are projected:

	Rate of return	Risk index
Bohemian Beer share	12 %	0.07
Moravian Wine share	9 %	0.09
Moravian Brandy share	15 %	0.05
Bohemian Milk share	7 %	0.03
Government bond	6 %	0.01

Tab. 2.3 Shares and Bond Evaluation

At the meeting of managers these restrictions have been appointed:

a) No more than 200 000 CZK might be invested in Bohemian Milk shares.
b) Government bonds should cover at least 20% of all investments.

c) Because of diversification of portfolio neither alcohol-drink company should receive more than 800 000 CZK.
d) Risk index of the final portfolio should be maximally 0.05.

Satisfying all the restrictions the management intends to maximize the annual return of the portfolio.

Formulation

1. Define the decision variables

Each variable represents amount of money invested in shares/bonds (in CZK):

x_1 = amount invested in Bohemian Beer shares,
x_2 = amount invested in Moravian Wine shares,
x_3 = amount invested in Moravian Brandy shares,
x_4 = amount invested in Bohemian Milk shares,
x_5 = amount invested in government bonds.

2. The objective function corresponds to maximizing the total annual return of portfolio

Since a value of x_1 is money invested in Bohemian Beer shares, the annual return ensuing from this investment is $0.12x_1$. The annual returns for other investments are counted similarly and the objective function (total annual return of portfolio) can be expressed as follows:

$$\text{Maximize } z = 0.12x_1 + 0.09x_2 + 0.15x_3 + 0.07x_4 + 0.06x_5 .$$

3. The constraints

a) No more than 200 000 CZK might be invested in Bohemian Milk shares:

$$x_4 \leq 200\,000 .$$

b) Government bonds should cover at least 20% of all investments:

$$x_5 \geq 400\,000 .$$

c) None of the alcohol-drink companies should receive more than 800 000 CZK:

$$x_1 \leq 800\,000 ,$$
$$x_2 \leq 800\,000 ,$$
$$x_3 \leq 800\,000 .$$

d) Risk index of the final portfolio should be maximally 0.05:

$$\frac{0.07x_1 + 0.09x_2 + 0.05x_3 + 0.03x_4 + 0.01x_5}{2\,000\,000} \leq 0.05 .$$

e) Budget restriction has to be respected:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2\,000\,000 .$$

4. The nonnegativity constraints

$$x_1, x_2, x_3, x_4, x_5 \geq 0 .$$

Optimal Solution

$x_1 = 800\,000$,
$x_2 = 0$,
$x_3 = 800\,000$,
$x_4 = 0$,
$x_5 = 400\,000$.

$z = 240\,000$.

Considering the definition of decision variables, the company should invest 800 000 CZK in Bohemian Beer shares, 800 000 CZK in Moravian Brandy shares and 400 000 CZK in government bonds. The expected annual return of the portfolio is 240 000 CZK.

2.5.5 Cutting Stock Problem

In some real situations, it is necessary to cut raw products into final products. The raw products are usually rolls of paper or textile, wooden sticks or bars, steel plates, etc. The goal is to get the final products with minimal loss of material. Alternative objective can be minimizing the number of cut or sliced raw products. If the final products are being assembled (as in following example) into a product that should be sold, the objective is usually the number of the assembled products. First, all possible cutting patterns must be found out (in practical applications the number of cutting patterns can be extremely large, but always finite). Each pattern corresponds to a decision variable, the value of which gives a number of raw products being cut according to the corresponding pattern.

Example 2.4

The company John Starling & Sons produces bird tables and bird boxes. The producer decided to prepare a special collection for an exhibition (with possible sales) that should be held in 20 days. The price of the bird table is set to 260 CZK; the price of the bird box is 570 CZK. Material and time requirements for assembling both the products can be found in Table 2.4. There is available stock of raw boards: 500 boards of length 1.1 m and 150 boards of length 1.4 m. These boards must be cut into final boards of length 30 cm and 25 cm. Available stock of screws is 3000 pieces. The producer can work 8 hours per day and intends to maximize the total revenue ensuing from the sales (all production is supposed to be sold).

	Bird table	Bird box
Wooden board 30 cm (pieces)	1	2
Wooden board 25 cm (pieces)	1	4
Screw (pieces)	8	16
Time (min)	30	60

Tab. 2.4 Production Inputs

Formulation

1. Define the decision variables

First of all, the producer has to find all the cutting patterns both for boards of length 1.1 m and for boards of length 1.4 m. Each pattern corresponds to the possible combination of final boards (30 cm and 25 cm) that can be cut from the larger boards (1.1 m and 1.4 m).

If, for example, we want to get only 30 cm boards from 1.1 m board, we obtain three of these boards (the loss is 20 cm in this pattern). If we decreased the number of 30 cm boards from three to two, would get two 25 cm boards. There is no loss in this pattern. All possible patterns are defined in the following table:

	1.1 m Board				1.4 m Board				
Pattern No	1	2	3	4	5	6	7	8	9
30 cm Board	3	2	1	0	4	3	2	1	0
25 cm Board	0	2	3	4	0	2	3	4	5
Loss (cm)	20	0	5	10	20	0	5	10	15

Tab. 2.5 Cutting Patterns

The boards of length 1.1 m can be cut with 4 different possibilities, 1.4 m boards with 5 possibilities. Hence, there are 9 decision variables in the model:

x_1 = number of 1.1 m boards cut according to the pattern number 1,
x_2 = number of 1.1 m boards cut according to the pattern number 2,
...
x_4 = number of 1.1 m boards cut according to the pattern number 4.

x_5 = number of 1.4 m boards cut according to the pattern number 5,
x_6 = number of 1.4 m boards cut according to the pattern number 6,
...
x_9 = number of 1.4 m boards cut according to the pattern number 9.

Two additional variables correspond to the products:

x_{10} = number of assembled bird tables,
x_{11} = number of assembled bird boxes.

2. The objective is to maximize the total revenue

$$\text{Maximize } z = 260x_{10} + 570x_{11}.$$

3. The constraints

a) 500 boards of length 1.1 m are available:

$$x_1 + x_2 + x_3 + x_4 \leq 500.$$

b) 150 boards of length 1.4 m are available:

$$x_5 + x_6 + x_7 + x_8 + x_9 \leq 150.$$

c) Available stock of screws is 3000 pieces:

$$8x_{10} + 16x_{11} \leq 3000 .$$

d) The producer can work 8 hours in 20 days (160 hours):

$$0.5x_{10} + x_{11} \leq 160 .$$

e) Number of obtained 30 cm boards must be greater than or equal to the number of 30 cm boards used for assembling (parameters are taken from Table 2.5):

$$3x_1 + 2x_2 + x_3 + 4x_5 + 3x_6 + 2x_7 + x_8 \geq x_{10} + 2x_{11} ,$$

or

$$3x_1 + 2x_2 + x_3 + 4x_5 + 3x_6 + 2x_7 + x_8 - x_{10} - 2x_{11} \geq 0 .$$

f) Number of obtained 25 cm boards must be greater than or equal to the number of all 25 cm boards used for assembling:

$$2x_2 + 3x_3 + 4x_4 + 2x_6 + 3x_7 + 4x_8 + 5x_9 \geq x_{10} + 4x_{11} ,$$

or

$$2x_2 + 3x_3 + 4x_4 + 2x_6 + 3x_7 + 4x_8 + 5x_9 - x_{10} - 4x_{11} \geq 0 .$$

4. The nonnegativity constraints and the integer type of all variables

$$x_1, x_2 , ..., x_{11} \geq 0 ,$$

$x_1, x_2 , ..., x_{11}$ are integers.

Optimal Solution

$x_1 = x_3 = x_4 = x_6 = x_7 = x_8 = x_{10} = 0,$
$x_2 = 65,$
$x_5 = 48,$
$x_9 = 102,$
$x_{11} = 160.$

z = 91 200 CZK.

If the company John Starling & Son wants to maximize the revenue, it should produce 160 bird boxes. The large boards will be cut according to the patterns number 2, 5 and 9. It is possible to show that this combination of used patterns is not the unique optimal solution.

2.5.6 Transportation Problem

Transportation problem deals with shipments (of material, goods, people, etc.) from a number of sources to a number of destinations. Each source has typically the limited supply and each destination has usually known demand. Unit shipping cost between each source and destination is defined. The objective is to find a feasible shipping schedule (shipped quantities) with minimal total shipping cost.

If the total supply equals the total demand, the model is called *balanced transportation model*. In real-life situations the problem is usually *unbalanced*, i.e. the total demand exceeds the total supply, or the total supply exceeds the total demand. Unbalanced models can be transformed into balanced by introducing a *dummy source* or *destination*. While shipments from a dummy source correspond to unsatisfied requirements, shipments to a dummy destination represent remains of suppliers.

Example 2.5

Star Chips, Inc. operating in the Czech Republic is going to establish three subsidiaries producing chips. They should be located in following cities: Benešov, Jihlava and Tábor. The main ingredient - potatoes - would be supplied from two warehouses in Humpolec and Pelhřimov (see Figure 2.17).

Fig. 2.17 Star Chips‘ Distribution Problem

The management of the corporation has estimated the weekly requirements of the companies. The warehouses’ capacities are limited. Potatoes are transported once a week from suppliers to destinations by train and it is possible to evaluate a unit shipping cost per ton. All the values are given in the following table:

Destination / Warehouse	Benešov	Jihlava	Tábor	Weekly Supply [t]
Humpolec	330	250	350	70
Pelhřimov	300	240	250	80
Weekly Demand [t]	45	60	35	140 / 150

Tab. 2.6 Supply, Demand and Unit Shipping Cost

The objective is to determine such deliveries from warehouses to destinations that minimize the total shipping cost. This shipping schedule, of course, must satisfy requirement of each destination, and must not exceed supply of any warehouse.

Formulation

As it is apparent from Table 2.6 the problem is unbalanced - total demand (140 tons) is less than the total supply (150 tons). The balanced model can be obtained after introducing a dummy destination with demand 10 tons (computed as the difference between the values 150 and 140):

Destination / Warehouse	Benešov	Jihlava	Tábor	Dummy	Weekly Supply [t]
Humpolec	330	250	350	0	70
Pelhřimov	300	240	250	0	80
Weekly Demand [t]	45	60	35	10	150 / 150

Tab. 2.7 Balanced Transportation Model

Since the dummy destination does not exist, no physical shipping occurs and therefore all the corresponding unit transportation costs equal zeroes.

1. Define the decision variables

In distribution models, it is reasonable to define the variables with two indices (generally x_{ij}). In case of transportation model the variable x_{ij} corresponds to the amount transported from the source i to the destination j.

x_{11} = amount of potatoes transported from Humpolec to Benešov [in tons],
x_{12} = amount of potatoes transported from Humpolec to Jihlava,
x_{13} = amount of potatoes transported from Humpolec to Tábor,
x_{14} = amount of potatoes remaining in Humpolec.

x_{21} = amount of potatoes transported from Pelhřimov to Benešov,
x_{22} = amount of potatoes transported from Pelhřimov to Jihlava,
x_{23} = amount of potatoes transported from Pelhřimov to Tábor,
x_{24} = amount of potatoes remaining in Pelhřimov.

2. The objective is to minimize the total shipping cost

If parameter c_{ij} is generally a unit shipping cost corresponding to the source i and the destination j, then $c_{ij}\,x_{ij}$ is the total shipping cost. If we evaluate the total shipping cost corresponding to all the sources and destinations, their sum gives the total shipping cost for our transportation problem. Thus, the objective function can be expressed as follows:

$$\text{Minimize } z = 330x_{11} + 250x_{12} + 350x_{13} + 300x_{21} + 240x_{32} + 250x_{33}\,.$$

3. The constraints

Real shipping must not exceed supply of warehouses and must satisfy demand of destinations. In the example, there are 6 constraints:

$$\begin{array}{llllllll l}
x_{11} & + x_{12} & + x_{13} & + x_{14} & & & & & = 70 \\
& & & & x_{21} & + x_{22} & + x_{23} & + x_{24} & = 80 \\
x_{11} & & & & + x_{21} & & & & = 45 \\
& x_{12} & & & & + x_{22} & & & = 60 \\
& & x_{13} & & & & + x_{23} & & = 35 \\
& & & x_{14} & & & & + x_{24} & = 10
\end{array}$$

4. The nonnegativity constraints

$$x_{11}, x_{12}, ..., x_{24} \geq 0 .$$

Optimal Solution

In Table 2.8 the optimal solution is shown. Each week, 60 tons is delivered from Humpolec to Jihlava. From Pelhřimov, 45 tons are transported to Benešov and 35 tons to Tábor. This shipping schedule minimizes the total weekly cost at the level 37 250 CZK. In Humpolec, 10 tons remain. This amount can be used for another purpose or the management should consider increase in the production of the subsidiaries.

Destination / Warehouse	Benešov	Jihlava	Tábor	Dummy	Weekly Supply [t]
Humpolec	-	60	-	10	70
Pelhřimov	45	-	35	-	80
Weekly Demand [t]	45	60	35	10	150 / 150

Tab. 2.8 Optimal Solution – Transported Amount

2.5.7 Assignment Problem

Consider the situation where some objects (service teams, jobs, employees, projects, etc.) should be assigned on a one-to-one basis to other objects. Each assignment can bring profit or incur cost. The objective is to maximize the total profit or minimize the total cost ensuing from the assignment structure.

Example 2.6

Prague Build, Inc. gets four commissions for building family houses in various parts of Prague (Michle, Prosek, Radlice, Trója). In the first step the company must solve the problem of excavating the shafts for basements. Each excavation takes 5 days. Management of the company decided to use four own excavators stored in four separated garages. The objective is to allocate each excavator to exactly one excavation with minimal cost. Since the costs are derived from distances [in km] between garages and destinations (Table 2.9), we can concentrate only on these distances to define the objective.

Destination / Garage	Michle	Prosek	Radlice	Trója
Garage 1	5	22	12	18
Garage 2	15	17	6	10
Garage 3	8	25	5	20
Garage 4	10	12	19	12

Tab. 2.9 Distances Between Garages and Destinations

Formulation

As you can see, the structure of assignment problem is rather similar to the structure of the transportation problem. However, here is one basic difference: all supplies and demands are equal to 1. Thus, exactly one excavator leaves each garage and each destination needs exactly one of the machines.

1. Define the decision variables

Whereas in transportation models the variable x_{ij} corresponds to the amount transported from the source *i* to the destination *j*, in assignment model this variable acts as an indicator of assignment of the object *i* to the object *j*. This *binary variable* equals 1 in case of assignment (matching) and 0 otherwise. In the example, $x_{ij} = 1$ if the excavator from the garage *i* goes to the destination *j*, and $x_{ij} = 0$ if the excavator from the garage *i* does not go to the destination *j*. It is quite evident that there are 16 bivalent variables in the example:

$$x_{ij} = 1 \text{ or } 0, \quad i, j = 1, 2, 3, 4 .$$

2. The objective is to minimize the total distance necessary for all excavators' movement

If parameter c_{ij} is a distance between the garage *i* and the destination *j*, the objective function can be expressed as follows:

$$\text{Minimize } z = \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij} \ ,$$

or using real unit cost:

$$\text{Minimize } z = 5x_{11} + 22x_{12} + ... + 12x_{44} .$$

The objective function corresponds to the sum of kilometers that all machines go together.

3. The constraints

From each garage exactly one excavator goes away:

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

Each destination needs exactly one excavator:

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + x_{24} + x_{34} + x_{44} = 1$$

Notice that the model is *balanced*, because the number of garages is equal to the number of destinations. Unbalanced model would need one or more dummy objects to become the balanced model.

Optimal Solution

In Table 2.10 the optimal assignment is dark shaded. The excavators go from Garage 1 to Michle, from Garage 2 to Trója, from Garage 3 to Radlice and from Garage 4 to Prosek. All machines go together 32 km (5 + 10 + 5 + 12). If the excavators return every day to their garages, the total daily distance is 64 km and in 5 days all machines would go 320 km. This value can be a basis for evaluating the total transportation cost.

Destination / Garage	Michle	Prosek	Radlice	Troja
Garage 1	5	22	12	18
Garage 2	15	17	6	10
Garage 3	8	25	5	20
Garage 4	10	12	19	12

Tab. 2.10 Optimal Assignment

2.6 Glossary

Assignment Problem – přiřazovací problém
A linear programming problem having a special structure, in which each of *n* resources must be assigned to each of *n* activities, on a one-to-one basis.

Balanced Transportation Problem – vyrovnaný dopravní problém
A transportation problem in which the total amount available at the suppliers exactly satisfies the total amount of requirements at destinations.

Binary Variable (Zero-One Variable) – bivalentní proměnná
Logical variable that must take a value of either 0 or 1.

Binding Constraint – omezení splněné jako rovnost
An inequality constraint for which, at a particular feasible point, left-hand side equals the right-hand side. Slack/surplus variable is equal to zero.

Blending Problem – směšovací problém
One of major types of linear programming problems. Typically, the final blend is mixed from a variety of sources having specific composition and price. The objective is usually to minimize the total cost, while keeping the required attributes of the final blend (quality, structure, weight, volume, content, etc.).

Constraints – omezení
Restrictions on the problem solution arising from limited resources, policy requirements, etc. An equation or inequality that rules out certain combinations of values for the variables involved.

Corner (Extreme) Point – krajní bod; vrchol
Feasible solution point occurring at the vertices of the feasible area (convex polyhedron). In two dimensions, corner point is determined by the intersection of two constraint lines.

Cutting Stock Problem – rozkrajovací problém; řezná úloha; úloha o optimálním dělení materiálu
One of major types of linear programming problems. The material must be cut into smaller portions to attain the specific goal.

Decision Variable – strukturní proměnná
Variable that represents managerial decision or component of decision that must be made in a problem solving situation.

Dummy Destination – fiktivní odběratel
A fictitious destination added to the unbalanced transportation problem in which supply availabilities exceed requirements. Dummy destination makes the problem balanced.

Dummy Source – fiktivní dodavatel
A fictitious source added to the unbalanced transportation problem in which requirements exceed supply availabilities. Dummy source makes the problem balanced.

Equation – rovnice
A restriction on some combination of the variables that must be equal to a particular value.

Feasible Area (Region) – množina přípustných řešení
The solution space or region that satisfies all the constraints simultaneously.

Feasible Solution – přípustné řešení
A solution that satisfies all the constraints.

Inequality Constraint – nerovnice
A restriction on some combination of the variables that must be greater than or equal to ($\geq$), or less than or equal to ($\leq$) a particular value.

Infeasible Solution – nepřípustné řešení
A solution that violates at least one constraint.

Linear Function – lineární funkce
A mathematical expression in which all the variables appear in separate terms and raise to the first power.

Linear Programming – lineární programování
A mathematical procedure for optimizing the linear objective function, respecting the set of linear constraints.

Linear Programming Model – model úlohy lineárního programování
A mathematical model with a linear objective function, a set of linear constraints, and nonnegative variables.

Maximization – maximalizace
Optimization of objective function, which looks for the highest objective value (e.g. profit).

Minimization – minimalizace
Optimization of objective function, which looks for the lowest objective value (e.g. cost).

Multiple (Alternative) Optimal Solutions – alternativní optimální řešení; nekonečně mnoho optimálních řešení
The existence of more than one optimal solution. The infinite number of optimal solutions.

Nonbinding Constraint – omezení splněné jako ostrá nerovnost
An inequality constraint for which, at a particular feasible point, left-hand side is less/greater than the right-hand side. Slack/surplus variable is positive.

Nonnegativity Constraints – podmínky nezápornosti
The restriction specifying that decision variables in the model must be positive or zero.

Objective Function – účelová funkce
A mathematical function expressed in terms of decision variables, which is to be optimized (maximized or minimized).

Objective Function Coefficients – koeficienty účelové funkce
Deterministic parameters in the objective function that evaluate the usefulness of the individual variables regarding the goal of the optimization.

Optimal (Optimum) Solution – optimální řešení
A feasible solution that maximizes or minimizes the objective function. The best of all feasible solutions.

Optimal (Optimum) Value – optimální hodnota účelové funkce
The value of the objective associated with the optimal solution.

Right-Hand Side Value – hodnota pravé strany omezení
A deterministic value on the right-hand side of the constraint.

Sensitivity Analysis – analýza citlivosti
An analysis that specifies the ranges for possible changes in original objective function coefficients and/or right-hand sides that have no "cardinal" impact on the reached optimal solution.

Simplex Method – simplexová metoda
An iterative algorithm for solving linear programming problems that only investigates corner points of the feasible area.

Slack Variable – přídatná proměnná
A nonnegative variable added to the left-hand side of "≤" constraint that converts it to equation.

Surplus Variable – přídatná proměnná
A nonnegative variable subtracted from the left-hand side of "≥" constraint that converts it to equation.

Transportation Problem – dopravní problém
A special type of linear programming problem that involves transportation or physical distribution of goods and services from suppliers to destinations.

Unbalanced Transportation Problem – nevyrovnaný dopravní problém
A transportation problem in which supply availabilities exceed requirements, or requirements exceed supply availabilities.

Unbounded Feasible Area – neomezená množina přípustných řešení
The feasible area extends infinitely. If the objective value is being improved in the direction of unboundedness, no optimal solution can be found.

Unique Optimal Solution – jediné optimální řešení
The only one solution of the managerial problem that maximizes/minimizes the objective function.

3. Network Models

As we mentioned in Section 1.3, some managerial problems can be described graphically as a *network*. It is a set of *nodes* and *arcs* that diagrams the relationships between objects of the real system. Network models can represent e.g. a transportation system where nodes are the cities and arcs are the connections between them (roads). The network expression is also suitable for description of computer systems (Local Area Network, World Wide Web, etc.), the company organization and production process, piping systems, or projects. In the following section there will be discussed the basic terminology and notation for network problems and some applications will be outlined.

3.1 Network Terminology

As we defined above, the network is a set of *nodes* and *arcs*. The arc is a connector between a pair of nodes and can be *directed* (*oriented*) or *undirected*. A directed arc indicates which node is considered as the point of origin. The orientation of the arc is marked with the arrowhead.

Fig. 3.1 Undirected and Directed Arc

If a network contains directed arcs it is called a *directed network*. Otherwise it is called an *undirected network*.

A *path* in a network is a specific sequence of arcs in which the initial node of each arc is identical with the terminal node of the preceding arc in the sequence. This sequence must cross the different nodes. In Figure 3.2 the example of such a path is shown (the sequence across the nodes 1 – 2 – 4 – 5 – 6).

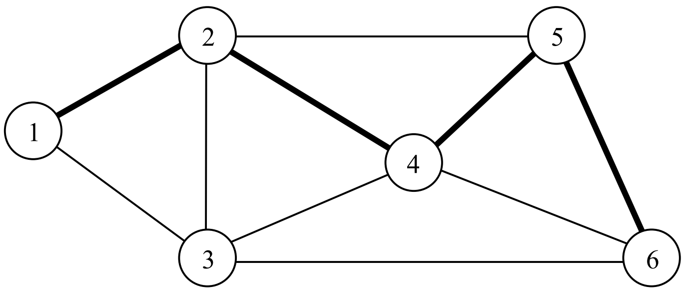


Fig. 3.2 Path Between Nodes 1 and 6

The path in Figure 3.2 is an *open path*, since it starts and ends in different nodes. If the path starts and ends in the same node (*closed path*) it is called a *circuit* (*cycle*).

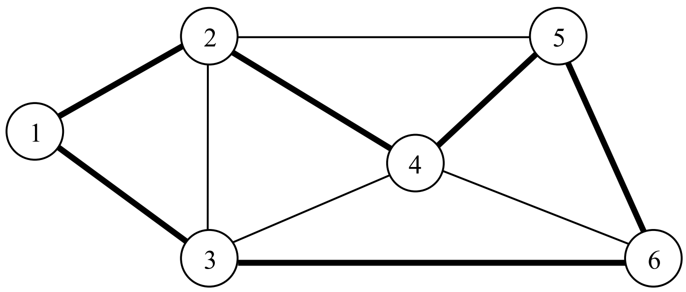


Fig. 3.3 Circuit

A network is called *connected* if there is a path connecting every pair of nodes in the network. The network shown above is connected. If we removed the arcs (1, 2) and (1, 3), we would no longer have the connected network.

The connected network without any circuit is called a *tree*. If we denote the total number of nodes in the network as n it is evident that the tree involving all nodes must contain exactly (n – 1) arcs. If we add any arc to the tree, the circuit will appear.

A *spanning tree* is a tree including all the nodes from the original network. The tree shown in Figure 3.4 is an example of spanning tree. Removing of any arc from the spanning tree changes the tree into *unconnected network*.

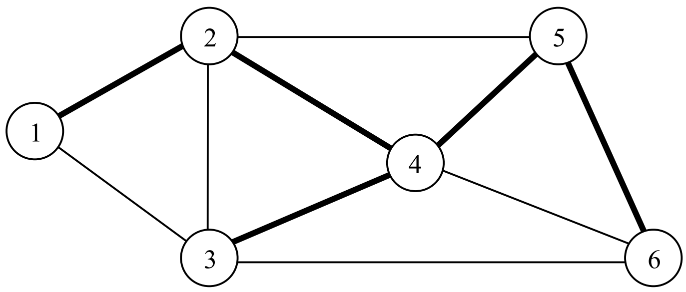


Fig. 3.4 Spanning Tree

In real situation the network is *evaluated*. The values can be added to nodes or/and to arcs and can represent time, distance, cost, capacity, etc. In the following section, several examples of network problems are described.

3.2 Basic Network Applications

Before describing the most often applications of network theory - project management - we mention some simple situations where the network representation is successfully used. With respect to the limited space in this textbook, it is not possible to describe all the methods used for solving problems and we will concentrate just on some simpler methods.

3.2.1 Shortest Path Problem

The problems in this category concern situations where we have to find the shortest path from an origin to a destination. Usually, there is no direct connection between these two points and the path crossing many other points must be found. A network represents the possible connections (arcs) between all points (nodes). Distance between the pair of connected nodes is attributed to each arc.

Example 3.1

In Figure 3.5 there is an example with 6 cities. Direct distances between the cities are in kilometers. The objective is to run from the city 1 to the city 6 through shortest path. It is very easy to find the path across the cities 2 and 3. The length of the shortest path is 40 km.

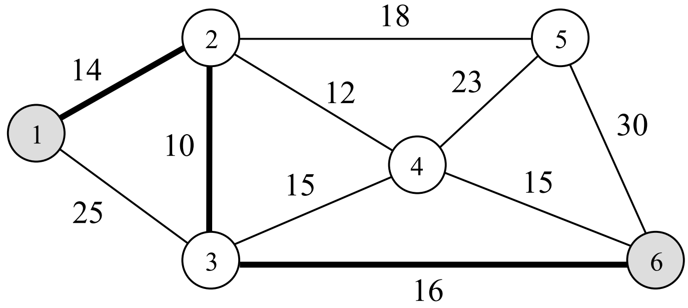


Fig. 3.5 Shortest Path Between Nodes 1 and 6

The problem can be formulated as finding the shortest paths not only from one origin to one destination, but from each node to all the other nodes (Table 3.1). Since the network is undirected the matrix of shortest paths is symmetric (e.g. the length of the path from 1 to 6 equals the length of the path from 6 to 1).

To / From	1	2	3	4	5	6
1	0	14	24	26	32	40
2	14	0	10	12	18	26
3	24	10	0	15	28	16
4	26	12	15	0	23	15
5	32	18	28	23	0	30
6	40	26	16	15	30	0

Tab. 3.1 Shortest Paths Between All Pairs of Nodes

3.2.2 Traveling Salesperson Problem (TSP)

Although this problem should be rather discussed in the section about integer programming, we outline it in this section because of its network representation.

Example 3.2

A salesperson has to visit a specified group of cities and come back to the origin (home city). This tour should be as short as possible in terms of the total distance. In Figure 3.6 such a tour (circuit) is displayed (it crosses the nodes 1, 2, 5, 4, 6, 3, 1). The optimal total distance is 111 km.

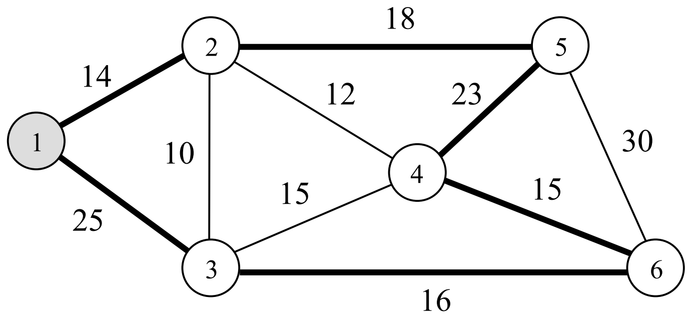


Fig. 3.6 Traveling Salesperson Problem

If we tabulate the direct distances between nodes, we can illustrate the similarity to the assignment problem, in which the goal is to mark 6 cells respecting the following rule: exactly one cell in each row and in each column is marked. Table 3.2 shows the optimal solution of TSP (dashes are used for unidentified connections).

From \ To	1	2	3	4	5	6
1	-	14	25	-	-	-
2	14	-	10	12	18	-
3	25	10	-	15	-	16
4	-	12	15	-	23	15
5	-	18	-	23	-	30
6	-	-	16	15	30	-

Tab. 3.2 Traveling Salesperson Problem

If the task had been solved as the classical assignment problem many solutions would have been found, infeasible in case of traveling salesperson problem. One of those solutions is strongly shaded in Table 3.3. When we graph this solution (Figure 3.7), we can see two *sub-tours* 1-2-3-1 and 4-5-6-4. Whereas it is feasible solution of assignment problem, it is infeasible in traveling salesperson problem. Therefore in formulation of the model, it is necessary to add constraints that prevent building sub-tours.

From \ To	1	2	3	4	5	6
1	-	14	25	-	-	-
2	14	-	10	12	18	-
3	25	10	-	15	-	16
4	-	12	15	-	23	15
5	-	18	-	23	-	30
6	-	-	16	15	30	-

Tab. 3.3 Assignment Problem

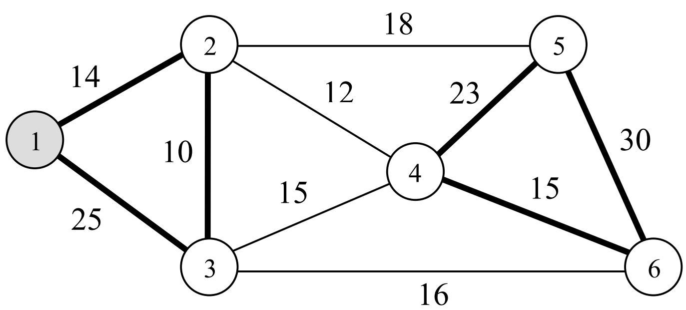


Fig. 3.7 Infeasible Sub-Tours in TSP

3.2.3 Minimal Spanning Tree

In some situations, instead of finding the shortest path or the shortest tour, the goal is to assure a connection between all nodes in the network. Assuming n nodes in the network, a s*panning tree* is such sub-network that contains exactly (n-1) arcs and no circuits. In case we have the evaluated network (e.g. by distances), the *minimal spanning tree* is a spanning tree with the minimal sum of values.

Example 3.3

We illustrate one of the possible algorithms on this elementary example. Suppose that the managerial problem is to connect 9 locations of an exhibition area with the source of electricity power. The objective is to minimize the cost of all the extensions. The direct distances (in meters) between locations can be found in Figure 3.8. The node 1 is the source of power. The price per 1 meter of a cable is 10 CZK.

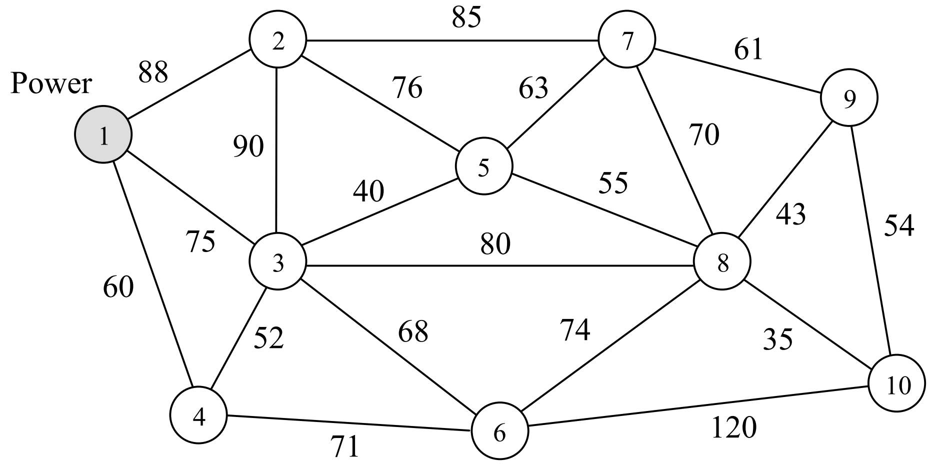


Fig. 3.8 Electricity Power Distribution in the Exhibition Area

Solution

1. In the first step we find two arcs with minimal distances. These are the arcs (8, 10) and (3, 5) with direct distances 35 and 40 meters (Figure 3.9).

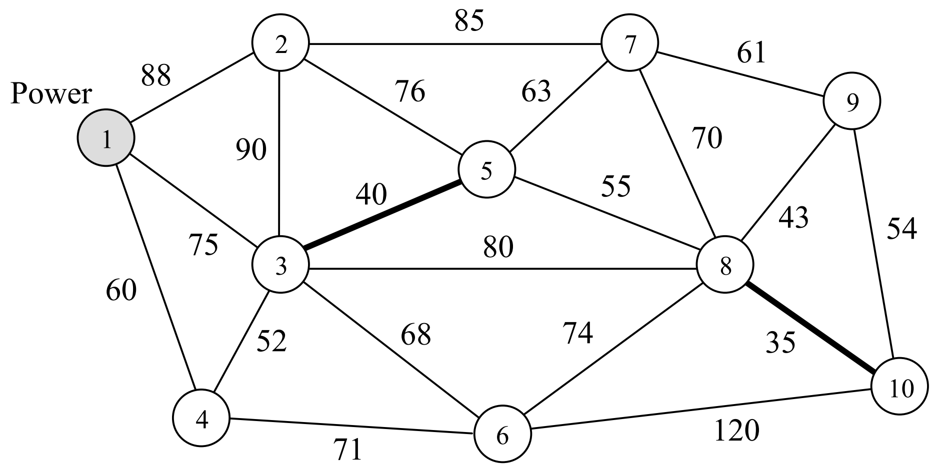


Fig. 3.9 First Step – Finding Two Arcs with Minimal Lengths

2. In the second step we search for another arc (from the set of all the remaining arcs in the network) with the minimal distance. This arc is added into the set of the arcs selected before. We must respect the important rule: no circuit can occur after adding a new arc. If the circuit appeared we would ignore this arc in the searching process.

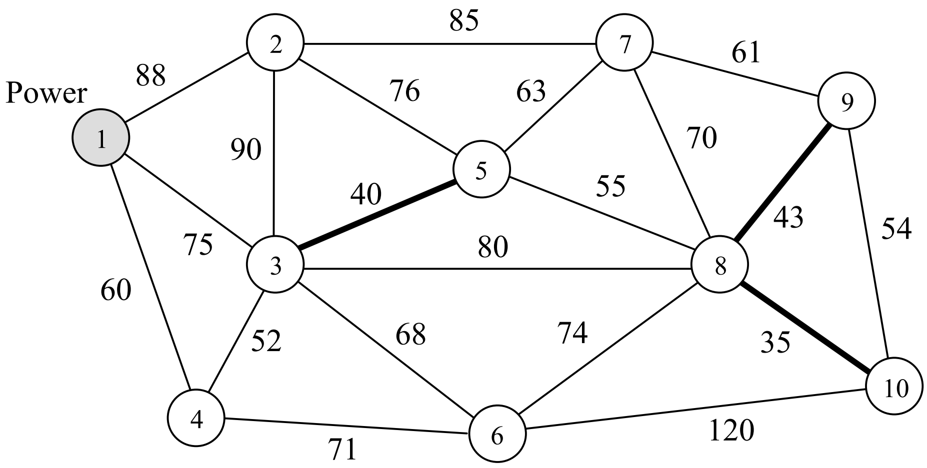


Fig. 3.10 Second Step – Adding a New Arc

3. We repeat the second step until a minimal spanning tree is found.

Figure 3.11 shows the optimal solution (minimal spanning tree). The total length of the cable necessary for all the extensions is 490 meters. Thus, total cost of distribution of electricity power in the exhibition area is 4900 CZK.

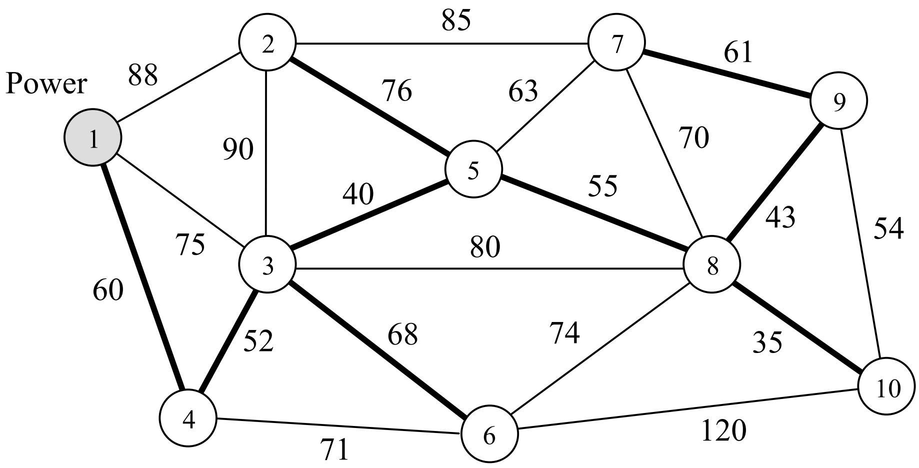


Fig. 3.11 Optimal Solution – Minimal Spanning Tree

3.2.4 Maximum Flow Problem

The objective is to find a maximal quantity of gas, fluid, traffic, information, people and so on that can be transported through a capacitated network. This network includes two important nodes – a starting node called the *source* and a terminal node called the *sink*. Whereas in directed networks the movement must respect the orientation of the arcs (movement in the reverse direction is restricted), in undirected networks movement in both directions is allowed.

There are five important rules that have to be considered during the construction of the mathematical model:

1. The quantity flowing through an arc must be less than or equal to the capacity of that arc.
2. For every node, except the source and the sink, the quantity flowing out of a node is equal to the quantity flowing into it.
3. Total quantity flowing into the source is zero.
4. Total quantity flowing out of the sink is zero.
5. Total quantities flowing out of the source and into the sink are equal.

Example 3.4

The board of White Lake City, which is situated on the edge of a small lake, solves the problem of minimizing disruptive effects of possible flood. One of effective solutions is to reconstruct the system of drains that would pump the water out of the lake into great reservoir. Since there are two alternatives of a drain system, the city board must decide which of them will be used.

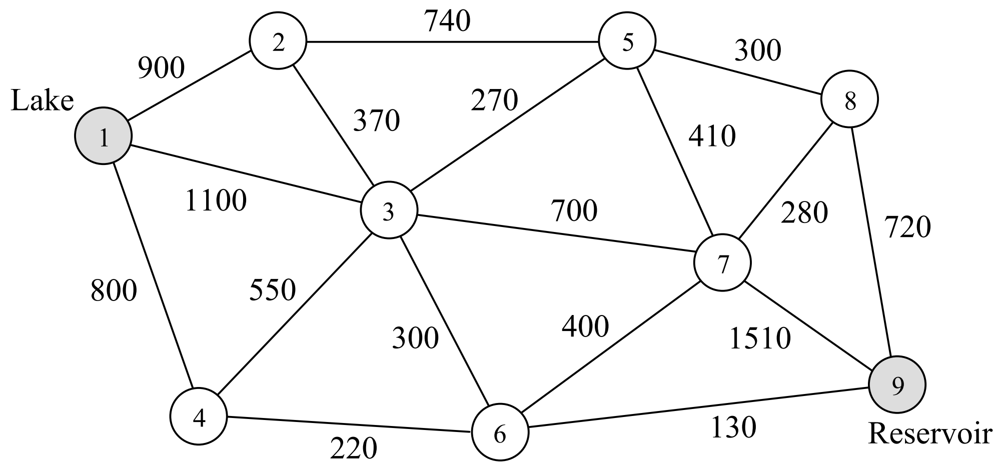


Fig. 3.12 Drain System – "Northern Channel"

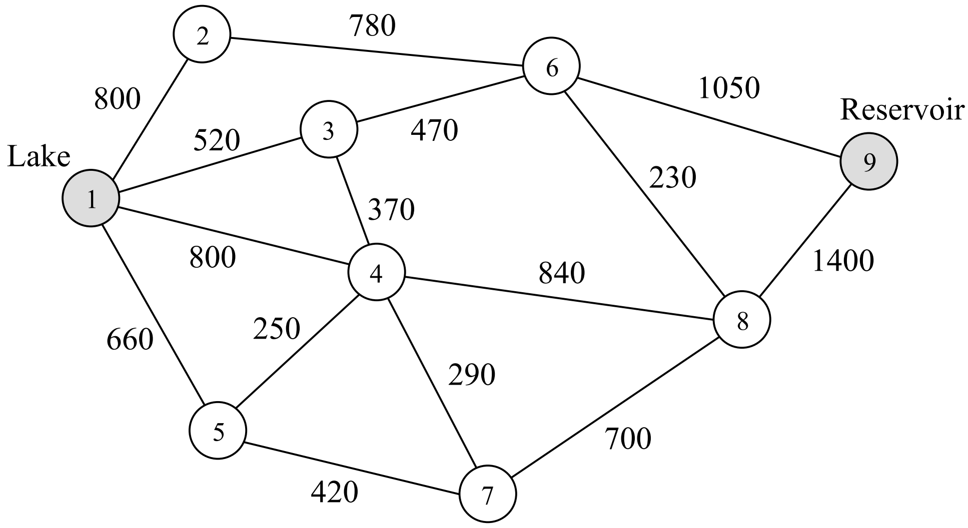


Fig. 3.13 Drain System – "Southern Channel"

The objective is to maximize the quantity of water being pumped in one hour. In both networks (Figures 3.12 and 3.13) the capacities of all the drains (in cubic meters per hour) are known.

Optimal Solution

If the city used the first alternative "Northern Channel", it would enable to pump maximally 1930 cubic meters per hour. Figure 3.14 shows for each drain the quantity of water and the direction in which the water will be flowing. Capacity of some drains is not saturated (e.g. (1, 2) or (6, 7)); some drains are not used at all ((2, 3), (3, 4) and (7, 8)).

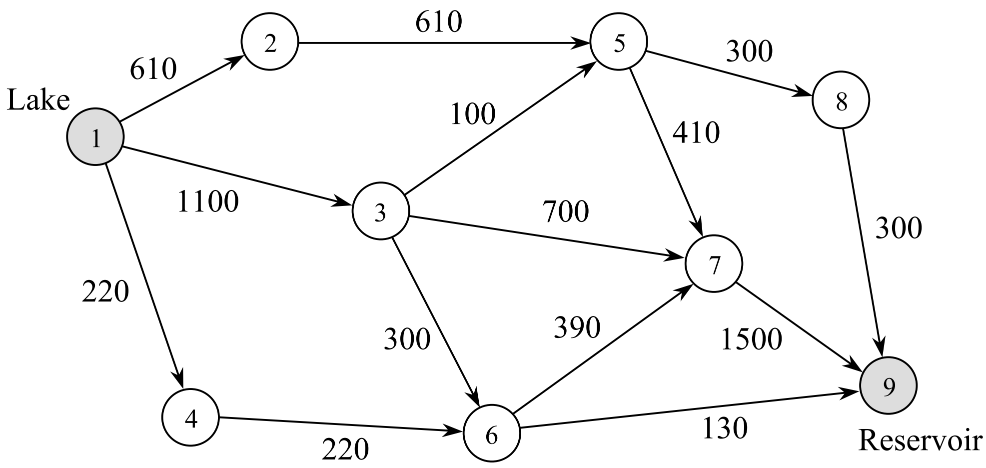


Fig. 3.14 Maximum Flow through "Northern Channel"

Note: A maximum flow problem has usually multiple optimal solutions. Using other algorithms could lead to another solution with the maximal flow 1930 cubic meters per hour, however with a different structure of individual flows.

In Figure 3.15 you can find the optimal solution for the second alternative "Southern Channel". Since the maximum flow equals 2270 cubic meters per hour, the city board should prefer this alternative to protect the White Lake City from the possible flood.

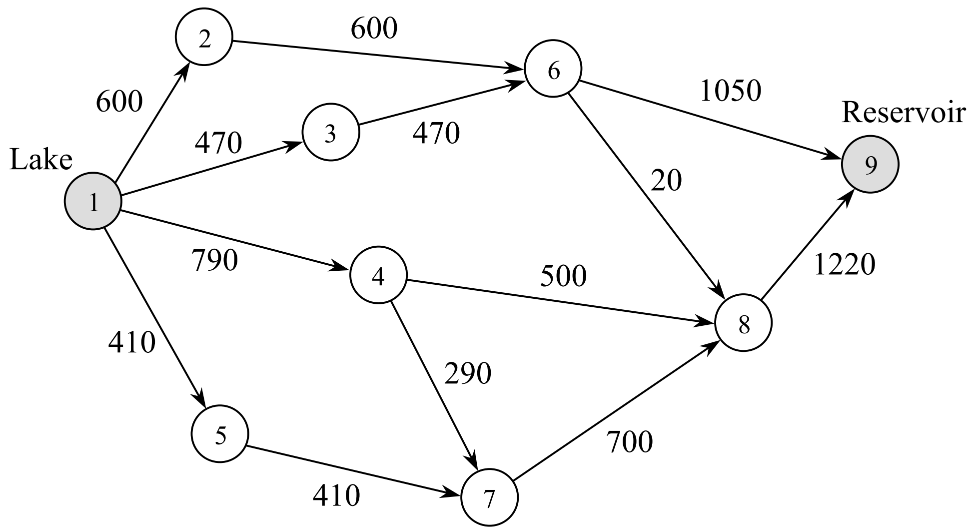


Fig. 3.15 Maximum Flow through "Southern Channel"

3.3 Project Management

Network models are used in the scheduling of large complex *projects* that consist of many interrelated *activities* (jobs, operations). All the activities take specific time and mostly require some resources for their completion that often correspond with specific cost. Each activity has a *start point* and a *finish point* (points in time) that are known as *events*. Since the activities are performed in a certain sequence, the final network must respect this restriction. In following text we will deal with the networks where arcs correspond to the activities and nodes correspond to the events.

Example 3.5

In Figure 3.16 you can find a simple example with two activities – designing a new model of car and making its prototype. Event 1 is the start point of Activity A. Event 2 can be considered as the finish point of Activity A and simultaneously as the start point of Activity B. Event 3 is the finish point of Activity B. The sequence of both activities is evident: the prototype cannot be made until the design of the car is finished (Activity A immediately *precedes* Activity B, or Activity B immediately *succeeds* Activity A).

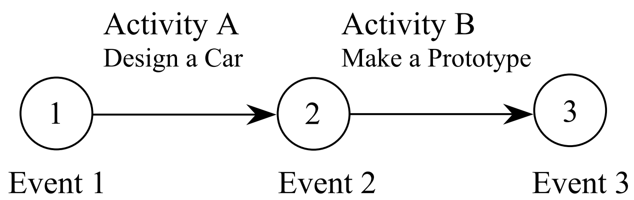


Fig. 3.16 Activities and Events

If we want to add a new activity (Computer Presentation) we must specify its *predecessors*. The public presentation of the car can start only after the finishing the designing works. However, this activity is independent of making the prototype. The network representing the existing activities is shown in Figure 3.17. Both making the prototype and computer presentation may start after the car has been designed.

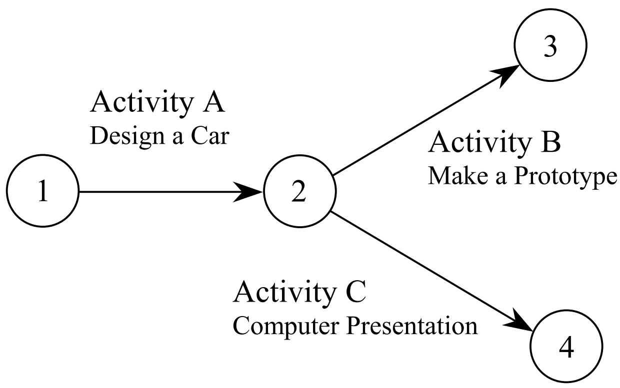


Fig. 3.17 Three Activities

The last two activities that we add in our illustrative example are Prototype Presentation and Prototype Test. Presentation of the prototype must, of course, succeed making the prototype and should follow the computer presentation. However, test of the prototype does not necessarily succeed computer presentation of the new model and it depends only on finishing the prototype. The only way how to express this requirement is introducing a *dummy activity* into the network (Figure 3.18). This fictitious activity assures that the activities and events are in proper sequence. As you can see from the figure, Activity D really follows both Activity B and Activity C, while Activity E succeeds only Activity B.

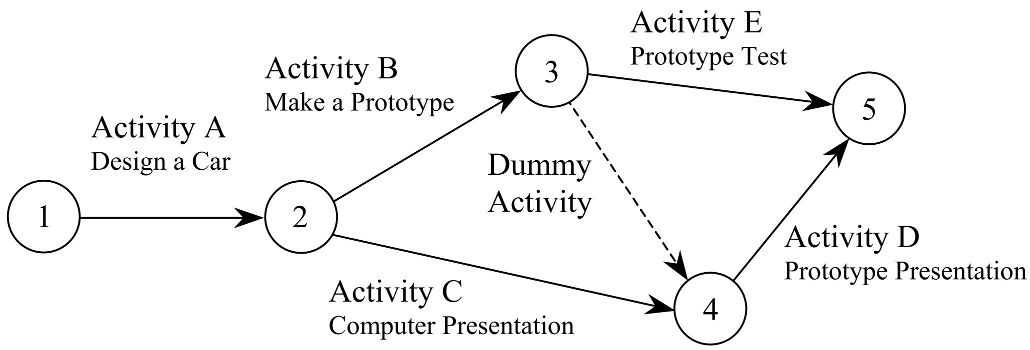


Fig. 3.18 Dummy Activity

Although a description of project can be easily understood, the construction of representative network could be rather difficult. The correct graphical expression of all interrelated activities is therefore the key step in project management. The sequence of all activities can be concisely described in the following table:

Activity	Activity Description	Immediate Predecessors
A	Design a Car	None
B	Make a Prototype	A
C	Computer Presentation	A
D	Prototype Presentation	B, C
E	Prototype Test	B

Tab. 3.4 New Car Project

A table like this can help especially the beginners to understand better the problem and construct the correct network step by step. The final step should be checking the constructed network according to the table. Extra relationships are inadmissible and no sequence can be lost.

In the following sections, we will be discussing time-oriented situations where the key issue of the project is its duration and scheduling of all the activities. For this purpose, two basic techniques were developed: *Critical Path Method* (CPM) and *Program Evaluation Review Technique* (PERT). From this point of view, there are three important phases of project management:

1. **Planning**. In this first phase, the entire project is decomposed into individual activities. It is necessary to determine the duration of each activity and its interrelation with other activities as it was shown in Example 3.5. The result of the planning phase is the constructed network as the graphical representation of the real project.

2. **Scheduling**. The objective of this phase is to propose, using CPM or PERT, start and finish times for each activity as well as the duration of the entire project. Some activities need a special attention because their possible delay causes delay of the whole project. Such activities are called *critical* and their sequence makes a *critical path*. Non-critical activities might be delayed; nevertheless an upper bound of possible delay must be computed.

3. **Controlling**. This is the final phase in project management. Managers should combine the network (as the result of the planning phase) and the time chart (as the result of the scheduling phase) to make periodic progress reports. Real performance of the project according to the proposed schedule requires everyday control and analysis. The network can be updated throughout time and a new schedule for the remaining part of the project should be then recalculated. Controlling phase is highly significant especially in case of large and long-term projects.

3.3.1 CPM

Critical Path Method (CPM), as a deterministic tool, deals with the projects where all activities are assumed to have exactly defined durations. As it has been said before the objective of the problem is determination of the following items:

1. Finish time of the project.
2. Start and finish times of each activity.
3. A critical path consisting of critical activities.
4. Non-critical activities and their possible delay.

The method is going to be explained on the following example.

Example 3.6
Project manager of direct marketing company Music DM, Inc. has received the direction to prepare the Christmas compilation of Czech carols. The offer should be sent to the address of company's customers, who are interested in carols. For this purpose, a thorough analysis should be carried out to select the best customers for the promotion. The project manager (PM) must consider the following activities to realize the project.

1. **Songs Selection**. In this first step the PM analyzes the questionnaires in which a sample of customers was asked about their interests. In cooperation with external music expert and experts on royalties the PM prepares a list of songs that will be compiled and defines medium sources of these songs.

2. **Mastering**. After the songs selection the CD can be mastered.

3. **Promotion Material Elaborating**. Promotion material is the offer being sent to each customer. It consists of the personal letter, the product presentation brochure (thus, songs selection must be a predecessor) and the order form. After elaborating the promotion material, the correction process must follow. In the example, these two steps are combined into one activity.

4. **Customers' Analysis**. This is a key step of the marketing process, because the right product must be offered to the right people. Precise analysis of customers' interests according to their previous behavior or their reactions on executed tests should lead to success of

the project. As the result of the analysis, the PM obtains the criteria for customers' selection and the estimated number of customers to be promoted.

5. **Promotion Material Production**. After promotion material is elaborated and the number of letters is estimated, the PM can order the production of all promotion materials.

6. **Make CD Copies**. Number of CD Copies is estimated according to the customers' analysis. Since only a segment of all promoted customers will be interested in the offered product, the PM makes a decision about the quantity of CDs to be produced.

7. **Customers Selection**. After the selection criteria are defined, the final selection of the best customers for the successful promotion can be approached.

8. **Data to Printing House**. All selected customers (their name and address) are sent to printing house.

9. **Promotion Material to Printing House**. Produced promotion material is sent to printing house where the individual letters will be printed.

10. **Laser Print**. Selected names and addresses are printed on the personal letters in the printing house. To simplify the example, all the materials, devoted to be sent to the customers, are supposed to be completed in the printing house. In practice these jobs carry out in different places.

11. **Mailing**. After all the promotion materials are put into envelopes, they are distributed to local post offices and then sent directly to final addressees.

All the envelopes have to be delivered until Thursday, December 12. The described activities together with their duration (in working days) and their immediate predecessors can be found in the following table.

Activity	Activity Description	Duration	Immediate Predecessors
A	Songs Selection	15	None
B	Mastering	8	A
C	Promotion Material Elaborating	6	A
D	Customers Analysis	7	A
E	Promotion Material Production	4	C, D
F	Promotion Material to Printing House	5	E
G	Customers Selection	3	D
H	Make CD Copies	12	B, D
I	Data to Printing House	3	G
J	Laser Print	9	F, I
K	Mailing	8	H, J

Tab. 3.5 Christmas Compilation Project

The PM must now set the starting date of the project such that it will be finished exactly on the specified mail date (December 12).

Solution

1. Construct the network

In the first step we have to construct a network that graphically represents the project (Figure 3.19). Two dummy (fictitious) activities D_1, D_2 have been added to assure correct interrelations between all real activities. Their durations are zeros.

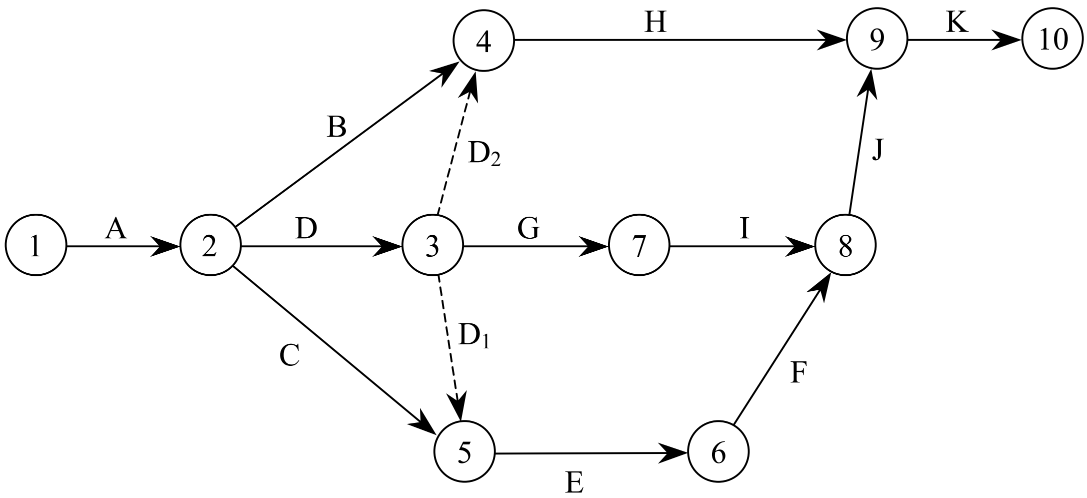


Fig. 3.19 Direct Marketing Project

While constructing the network, we should respect the following rules:

- A network must have one start node and one finish node.
- Each activity must be represented just by one arc.
- Two nodes are connected maximally by one arc.
- The node representing the completion of an activity has higher number than the node representing the start of this activity – the rule prevents circuits in the network.

2. Event analysis

For this purpose, we must introduce some new terms:

> The **earliest event time for a node** is the earliest time at which all the preceding activities have been completed. The **earliest event time for completing a project** is the earliest time of the last event.

This definition corresponds to the rule that all activities starting in a node can be realized after all predecessors have been completed.

> The **latest event time for a node** is the latest time at which the event can occur without delaying the determined completion time of the project. The **latest event time for completing a project** can be set in the planning process or can be considered as the earliest event time for completing a project.

All the earliest event times and the latest event times are computed in two phases: *forward pass* and *backward pass*.

Forward pass

We compute the earliest event times for all the nodes. The computation runs through the nodes according to their numbers (in ascending order), from the start of the project to its finish. For better understanding this phase we will compute all the times directly in the network. For this purpose, we divide each node into three parts (Figure 3.20).

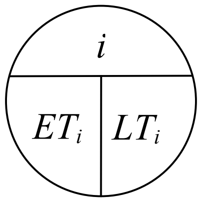


Fig. 3.20 Node Definition for Using CPM

In the top part of the node there is its number. In Figure 3.20 we consider a general node with the number i. Bottom part of the node is divided into the left and the right part. Whereas in the left part we will compute the earliest event time for the node (ET_i), in the right part we get the latest event time (LT_i).

Thus, in the first phase we must compute ET_i for each node in the network. Figure 3.21 shows the *forward pass* for the Example 3.6. We describe the process of computation in the following steps:

1. Set $ET_1 = 0$. This is the start time of the activity A (1,2) and of the project itself.

2. The earliest event time ET_2 determines the possible start of all the activities coming out from the node 2. All of them – D (2,3), B (2,4), C (2,5) – can start after all previous activities had been finished (in this case it is the only activity A (1,2)). Earliest time is 0 + 15 = 15. Thus, $ET_2 = 15$.

3. $ET_3 = 15 + 7 = 22$. The earliest event time of the node 3 is computed similarly to the node 2.

4. Since the node 4 represents the finish node of two activities – B (2,4) and D_2 (3,4) – the earliest event time ET_4 must be computed as the maximum of possible finish times of these activities: $ET_4 = \max(15 + 8, 22 + 0) = 23$.

   This rule can be generally expressed as follows:

$$\boxed{ET_j = \max_i (ET_i + t_{ij}),} \tag{3.1}$$

   where
   ET_j … earliest event time of the node j that is the finish node of the activity (i, j),
   ET_i … earliest event time of the node i that is the start node of the activity (i, j),
   t_{ij} … duration of the activity (i, j).

5. $ET_5 = \max(22 + 0, 15 + 6) = 22$.
6. $ET_6 = 22 + 4 = 26$.

7. $ET_7 = 22 + 3 = 25$.
8. $ET_8 = \max(26 + 5, 25 + 3) = 31$.
9. $ET_9 = \max(23 + 12, 31 + 9) = 40$.

10. $ET_{10} = 40 + 8 = 48$. This is the earliest event time for completing the project. It is evident that the entire project cannot be completed earlier than in 48 days. We denote this time as T.

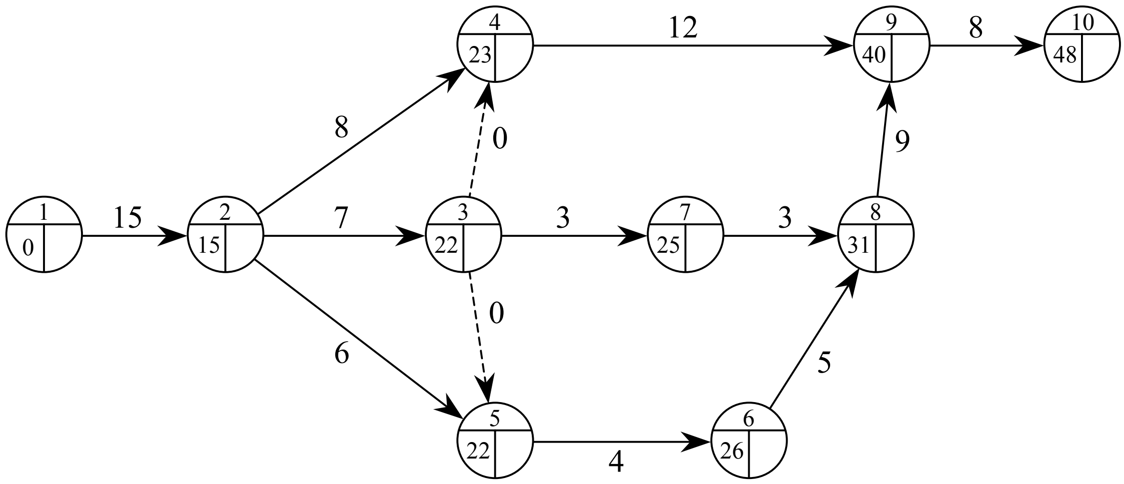


Fig. 3.21 CPM – Forward Pass

Note: The earliest event time for completing the project ET_{10} is determined by the longest path leading from the node 1 to the node 10.

Backward pass

This phase of CPM involves computing the latest event times for all the nodes. Whereas the forward pass crosses the nodes in ascending order according to their numbers, backward pass goes from the finish of the project to its start in the opposite (descending) order. *Backward pass* is shown in Figure 3.22.

1. First, it is necessary to set the latest event time for completing the project. There are two possibilities:

   a) We set $LT_{10} = ET_{10}$. The latest event time of completing the project equals to its earliest event time.

   b) $LT_{10} = T_{PL} > ET_{10}$. A duration of the entire project can be *planned* by the management of the company to T_{PL}.

   In the example, we select the first possibility $LT_{10} = ET_{10} = T$. The value of T corresponds to the shortest duration of the project.

2. The latest event time LT_9 determines the latest possible start of the activity K (9,10) without delaying the project. Since the activity duration is 8 days and the project must be completed in 48 days, $LT_9 = 48 - 8 = 40$.

3. Similarly, for the node 8, the latest event time is computed as $LT_8 = 40 - 9 = 31$. It ensues from the simple rule: if the project should not be delayed, all successive latest event times (computed before) must be satisfied. In such case, if the project should be completed in 48 days, event number 9 should occur in 40 days and thus, the activity J (8,9) must start at the latest in 31 days.

4. $LT_7 = 31 - 3 = 28$.
5. $LT_6 = 31 - 5 = 26$.
6. $LT_5 = 26 - 4 = 22$.
7. $LT_4 = 40 - 12 = 28$.

8. Computing the latest event time of the node 3 is not as easy as the previous computations. LT_3 depends not only on one successive latest event times, but even on three of them (those events correspond to the nodes 4, 5 and 7). Therefore the latest event time must be computed according to the following formula:

$$\boxed{LT_i = \min_j (LT_j - t_{ij})\,,} \tag{3.2}$$

where
LT_i … latest event time of the node i that is the start node of the activity (i, j),
LT_j … latest event time of the node j that is the finish node of the activity (i, j),
t_{ij} … duration of the activity (i, j).

Using the formula for the node 3, we get $LT_3 = \min(28 - 0, 22 - 0, 28 - 3) = 22$.

If we did not respect this rule and we computed, e.g. $LT_3 = 28 - 3 = 25$, it would not be possible to meet $LT_5 = 22$.

9. Similarly, $LT_2 = \min(22 - 7, 28 - 8, 22 - 6) = 15$.
10. $LT_1 = 15 - 15 = 0$.

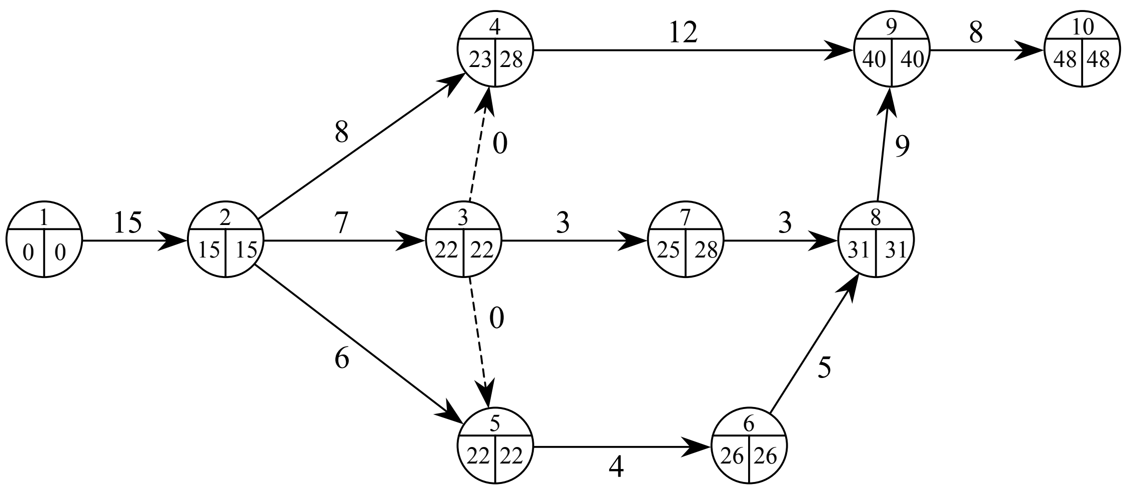


Fig. 3.22 CPM – Backward Pass

3. Activity analysis

After computing the earliest event times and the latest event times for all the nodes, we must approach the final step of the project analysis – determination of *critical* and *noncritical activities*. Critical activities are such activities, the delay of which causes delay in completion of the entire project. On the other hand, each noncritical activity can be delayed. However, its delay is limited. For this purpose, the *total float* (also called the *slack*) for each activity must be computed:

$$\boxed{TF_{ij} = (LT_j - ET_i - t_{ij})\,,} \tag{3.3}$$

where

TF_{ij} … total float of the activity (i, j),
LT_j … latest event time of the node j that is the finish node of the activity (i,j),
ET_i … earliest event time of the node i that is the start node of the activity (i, j),
t_{ij} … duration of the activity (i, j).

> Total floats of critical activities are zeroes.
> Total floats of noncritical activities are positive.

Note: If the planned duration of the project is considered in the backward pass as $T_{PL} > T$, total floats of all the critical activities are equal to $(T_{PL} - T)$.

A sequence of critical activities determines a *critical path* (this term is also used in the name of CPM – critical path method). The critical path is the longest path leading from the start of a project to its finish. In many projects multiple critical paths exist. As mentioned above, the determination of the critical path needs the computation of the total float for each activity. In Figure 3.23, these numbers are computed in brackets. The critical activities (with zero total floats) are highlighted.

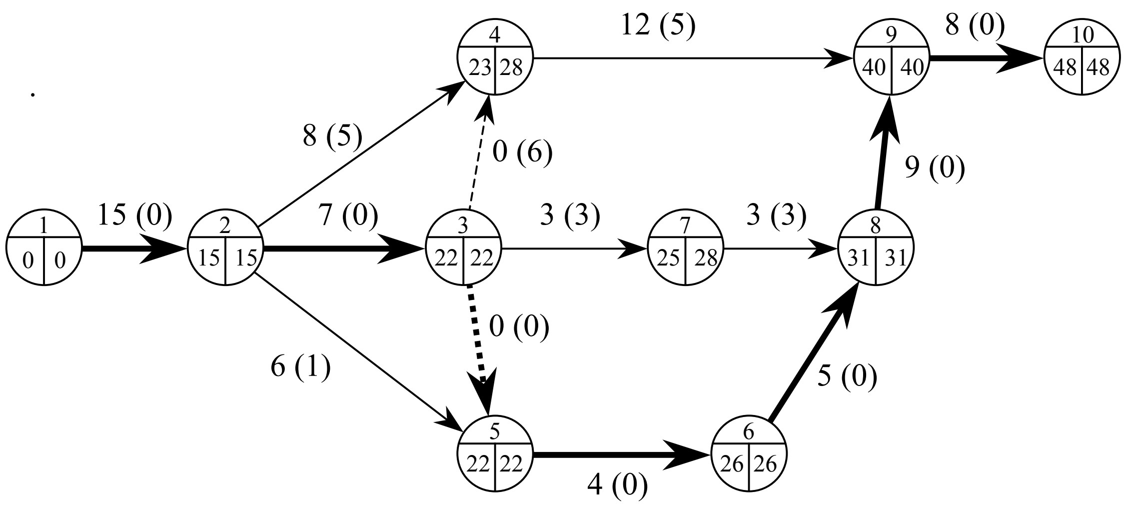


Fig. 3.23 CPM – Total Floats and Critical Activities

There is the only one critical path in the project consisting of following 6 critical activities (as the critical activity D_1 is dummy, it is not in the list):

Activity	Description of Activity	Earliest Start Time	Earliest Start Date	Latest Finish Date
A	Songs Selection	0	Oct 4	Oct 24
D	Customers Analysis	15	Oct 25	Nov 5
E	Promotion Material Production	22	Nov 6	Nov 11
F	Promotion Material to Printing House	26	Nov 12	Nov 18
J	Laser Print	31	Nov 19	Nov 29
K	Mailing	40	Dec 2	Dec 11

Tab. 3.6 Critical Activities

None of critical activities can be delayed if the project should be completed in T = 48 days. As the management decided to deliver all the envelopes until December 12, the project has to be started on October 4 (48 working days ago). It is evident that starting the project on October 3 causes increasing in all the total floats by 1 day (the planned project duration $T_{\mathrm{PL}} = 49$ days).

The positive total floats determine noncritical activities (the dummy activity D_2 is not in the list):

Activity	Description of Activity	Earliest Start Time	Earliest Start Date	Latest Finish Date	Total Float
B	Mastering	15	Oct 25	Nov 13	5
C	Promotion Material Elaborating	15	Oct 25	Nov 5	1
G	Customers Selection	22	Nov 6	Nov 13	3
H	Make CD Copies	23	Nov 7	Nov 29	5
I	Data to Printing House	25	Nov 11	Nov 18	3

Tab. 3.7 Noncritical Activities

Total floats give three possibilities of delay of each noncritical activity:

1. A start of the activity can be postponed.
2. Duration of the activity can be extended (the activity can be even interrupted).
3. The combination of 1 and 2.

All postponements and extensions, of course, must respect the total floats.

3.3.2 PERT

Whereas in CPM only one estimate of each activity's duration is considered, PERT (Program Evaluation Review Technique) uses three estimates to form a weighted average of the expected completion time:

a_{ij} = *optimistic estimate* of duration – the shortest possible time, in which the activity (i, j) can be accomplished (if execution goes extremely well),

b_{ij} = *pessimistic estimate* of duration – the longest time that the activity (i, j) could take (when everything goes wrong),

m_{ij} = *most likely estimate* of duration – the time that would occur most often (if the activity (i, j) were repeated under exactly the same conditions many times), or the time estimated by experts.

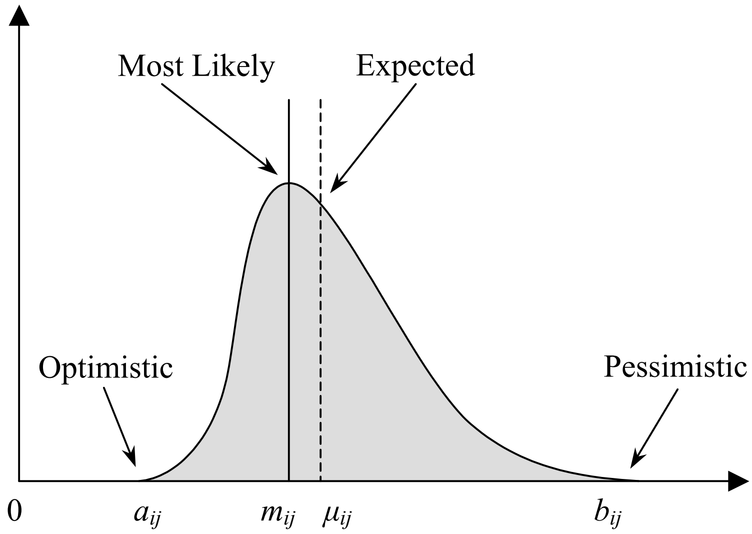


Fig. 3.24 Activity Time β-distribution

If we consider the β probabilistic distribution of activity time (Figure 3.24), the *expected completion time* of the activity (i, j) can be then computed as a weighted average of the three time estimates using the following formula:

$$\mu_{ij} = \frac{a_{ij} + 4m_{ij} + b_{ij}}{6}. \tag{3.4}$$

In addition it is possible to calculate the *standard deviation* for each activity:

$$\sigma_{ij} = \frac{b_{ij} - a_{ij}}{6}. \tag{3.5}$$

After computing the expected completion time μ_{ij} for all the activities, standard CPM is used ($t_{ij} = \mu_{ij}$) to determinate a critical path. Whereas in a deterministic model the value of T (the total duration of the project) can be computed as the sum of durations of the activities on the critical path (CP), in a probabilistic model we use their expected completion times for calculating the *expected duration of the project*:

$$M = \sum_{(i,j)\in \mathrm{CP}} \mu_{ij}. \tag{3.6}$$

Similarly, the *variance of the project duration* can be computed as follows:

$$\sigma^2 = \sum_{(i,j)\in \mathrm{CP}} \sigma_{ij}^2. \tag{3.7}$$

Real duration of the project is a continuous random variable with the mean M and the variance σ^2 (or the *standard deviation* σ). It is possible to prove (on the base of *central limit theorem*) that this random variable tends to be *normally distributed*.

Then, two simple managerial questions can be answered:

1. **What is the probability of project completion within a desired time T_D?**
   The normal distribution N (M, σ) can be transformed into the standard normal distribution N (0, 1) using the following formula:

$$z = \frac{T_D - M}{\sigma} . \tag{3.8}$$

   For determination of the required probability we just look up the value of the standard normal distribution corresponding to the value z.

2. **What is the completion time T_D in which the project will be finished with a desired probability p?**
   For this purpose, we look up the value z_p corresponding to the probability p and then we calculate the required completion time T_D as follows:

$$T_D = M + z_p \sigma . \tag{3.9}$$

Note: The table of the standard normal distribution values can be found in Appendix.

Example 3.7

To illustrate PERT, we modify Example 3.6 by introducing three duration estimates for each activity (Table 3.8).

Activity	Description of Activity	a_{ij}	m_{ij}	b_{ij}	μ_{ij}	Immediate Predecessors
A	Songs Selection	11	15	19	15	None
B	Mastering	7	8	9	8	A
C	Promotion Material Elaborating	5	6	7	6	A
D	Customers Analysis	5	7	9	7	A
E	Promotion Material Production	2	3	10	4	C, D
F	Promotion Material to Printing House	3	4	11	5	E
G	Customers Selection	2	3	4	3	D
H	Make CD Copies	8	11	20	12	B, D
I	Data to Printing House	2	3	4	3	G
J	Laser Print	6	8	16	9	F, I
K	Mailing	6	8	10	8	H, J

Tab. 3.8 Christmas Compilation Project – Three Estimations of Activities Duration

Solution

1. Construct the network

The considered change, of course, does not affect the network representing the original project in terms of the interrelations between all the activities (Figure 3.19).

2. Determine the expected completion time for each activity

All the expected completion times are computed using Equation 3.4. For example, the weighted average for the activity A (1,2) is:

$$\mu_{12} = \frac{11 + 4(15) + 19}{6} = 15 .$$

Similarly, we must determine these numbers for all the other activities. The estimations for each activity in Table 3.8 were intentionally set so as the computed values μ_{ij} equal to the duration values specified in Table 3.5, which enables us to compare the results (CPM and PERT).

3. Event analysis

This phase is identical with the computation in Example 3.6. In each step, of course, we use μ_{ij} instead of t_{ij}. Although Figure 3.22 shows the results of the computation, it is necessary to consider probabilistic feature of the project. All the computed numbers (the earliest and the latest event times) are actually expected values (averages) of the associated random variables. The expected duration of the project is $M = 48$ days.

4. Activity analysis

Using Equation 3.3 it is possible to compute *total float* for each activity, or rather its expected value. The activities with zero total floats are critical. The only one critical path (identical with the critical path in Example 3.6) consists of the following activities:

Activity	Description of Activity	a_{ij}	m_{ij}	b_{ij}	μ_{ij}	σ_{ij}
A	Songs Selection	11	15	19	15	8/6
D	Customers Analysis	5	7	9	7	4/6
E	Promotion Material Production	2	3	10	4	8/6
F	Promotion Material to Printing House	3	4	11	5	8/6
J	Laser Print	6	8	16	9	10/6
K	Mailing	6	8	10	8	4/6

Tab. 3.9 Critical Activities in PERT

Standard deviations σ_{ij} in Table 3.9 are computed using Equation 3.5 (the values are expressed in fractions).

Using formulas (3.6) and (3.7) we get the *expected duration* and the *variance* (*standard deviation*) *of duration* of the project:

$$M = 15 + 7 + 4 + 5 + 9 + 8 = 48 ,$$

$$\sigma^2 = (8/6)^2 + (4/6)^2 + (8/6)^2 + (8/6)^2 + (10/6)^2 + (4/6)^2 = 9 ,$$

$$\sigma = 3 .$$

As we mentioned above, a real duration of project is the random variable with (approximately) the normal distribution N (M, σ). Thus, in the example, it is the distribution N (48, 3).

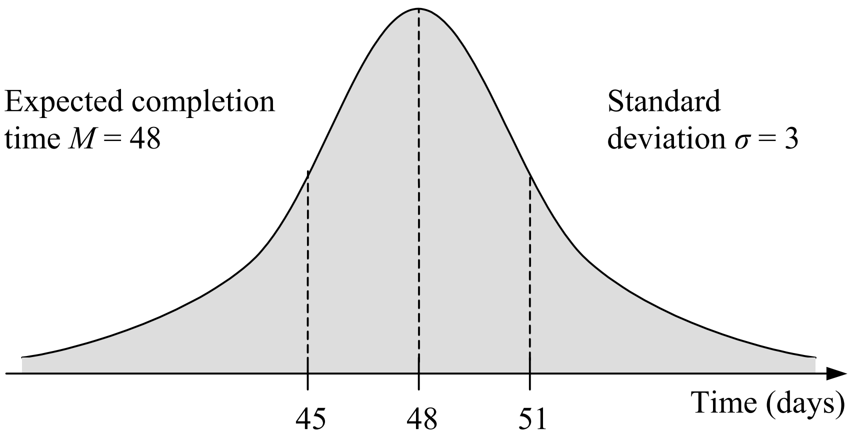


Fig. 3.25 Normal Distribution of the Project Completion Time

Let us suppose that the management of the company wants to find the probabilities of the project completion within 45 days (i.e. until December 9) and within 54 days (i.e. until December 20). First we compute transformed values of standard normal variables:

$$z_1 = \frac{45-48}{3} = -1,$$

$$z_2 = \frac{54-48}{3} = 2.$$

In the table of the standard normal distribution N (0, 1) we look up the distribution values corresponding to the required probabilities:

$$p_1 = 0.1587,$$

$$p_2 = 0.9772.$$

Hence, the probability with which the project will have been finished by December 9 is 0.1587. The probability of finishing the project by December 20 is 0.9772.

As mentioned above, the management, of course, might solve the opposite question: to find the date by which the project will have been finished, e.g. with a reliability of 95%. For this purpose, we must look up the value z_p corresponding to the probability $p = 0.95$. The value found in the table is $z_p = 1.645$. Introducing this value into Equation 3.9 we get:

$$T_D = 48 + 1.645\,(3) = 52.935.$$

If we round the result, we can take 53 days as the duration of the project corresponding to the desired probability. Thus, the project will have been finished (with the reliability of 95%) by December 19.

3.4 Glossary

Activity – činnost
A specific job or task that is part of a project and requires time and resources for its completion.

Arc – hrana
A connector between a pair of two nodes.

Circuit (Cycle) – cyklus
The path that starts and ends in the same node (closed path).

Connected Network – souvislý graf
A network that contains a path connecting every pair of nodes in the network.

CPM (Critical Path Method) – metoda hledání kritické cesty
A network-based scheduling procedure typically used for construction of projects.

Critical Activity – kritická činnost
An activity on a critical path. All of these activities have either zero total float or the same value of minimum total float in the CPM/PERT network.

Critical Path – kritická cesta
The longest path in the CPM/PERT network. Its length expresses the total time required to complete the project.

Directed Arc – orientovaná hrana
An arc with the specification of the point of origin. The orientation is marked with the arrowhead.

Directed Network – orientovaný graf
A network containing any directed arc.

Dummy Activity – fiktivní činnost
A fictitious activity that is used to maintain the predecessor relationships in the CPM/PERT network.

Duration – doba trvání činnosti
Time required to complete an activity.

Earliest Start Time – nejdříve možný začátek
The earliest time at which an activity may start.

Event – událost (začátek nebo konec činnosti)
Starting or ending point in time for an activity/activities.

Expected Completion Time – očekávaná doba trvání činnosti/projektu
Expected duration of an activity/entire project.

Latest Finish Time – nejpozději přípustný konec
The latest time at which an activity can be finished without delaying the entire project.

Maximal (Maximum) Flow – maximální tok
The distribution of flows through a network that permits the greatest amount to flow from the source to the sink.

Most Likely Estimate – modální odhad doby trvání činnosti
An estimate of an activity's duration that is considered to be the most likely (modal value).

Network – graf; síť
A set of nodes and arcs that diagrams the relationships between objects in the real system.

Node – uzel
A junction point in the network.

Optimistic Estimate – optimistický odhad doby trvání činnosti
An estimate of an activity's duration under ideal conditions. The shortest possible time necessary to complete the activity.

Path – cesta
A sequence of arcs in which the initial node of each arc is identical with the terminal node of the preceding arc in the sequence.

PERT (Program Evaluation Review Technique) – PERT
A planning and monitoring technique for projects based on three time estimates for duration of each activity.

Pessimistic Estimate – pesimistický odhad doby trvání činnosti
An estimate of an activity's duration under the worst conditions. The longest time necessary to complete the activity.

Project – projekt
A collection of interrelated activities. Each activity has a start point and a finish point.

Shortest Path – nejkratší cesta
A path from one point to another with the minimal sum of attributed values.

Sink – výstup
The point in the network where all flows end.

Source – vstup; zdroj
The point in the network where all flows begin.

Spanning Tree (of the Network) – kostra grafu
A tree including all the nodes of the original network.

Total Float (Slack) – celková časová rezerva
The extra time that an activity can be held up without delaying the project's completion.

Tree – strom
A connected network without any circuit.

Unconnected Network – nesouvislý graf
A network that contains at least one pair of nodes that cannot be connected with a path.

Undirected Arc – neorientovaná hrana
An arc without the specification of the point of origin.

Undirected Network – neorientovaný graf
A network that contains only undirected arcs.

4. Inventory Models

Inventory represents the goods or material that must be held by a company for use sometime in the future. The usual examples of inventories are:

- raw material,
- finished goods,
- semi-finished products,
- spare parts.

The key reasons for maintaining inventory are *protection against fluctuating demand*, *protection against delayed supply*, *savings on ordering cost*, *benefits on large quantities*, *protection against inflation*, etc. All these reasons (and many others) lead to following two questions for a company's management:

1. How much to order?
2. When to order?

To be able to answer these questions, the most important types of cost associated with inventory management must be considered in the appropriate analysis.

4.1 Inventory Terminology and Models Classification

Since the objective in the inventory model is typically the minimization of the total cost, we have to identify the most important types of **partial inventory cost**:

1. *Ordering and setup cost* are all the necessary expenses of placing an order. This item represents the fixed charge that includes e.g. the cost of paper work, billing cost and supplier's fixed cost associated with the order. More frequent ordering of smaller quantities results in a higher cost during the period than less frequent ordering of larger quantities.

2. *Unit purchasing cost* is the variable cost associated with purchasing a single unit. This item becomes an important factor in case the price of the single unit depends on the size of the order (quantity discount – unit price decreases with increasing order quantity).

3. *Holding or carrying cost* includes:

    - storage cost (maintaining the storage space),
    - store keeping operations (taking physical inventory),
    - insurance and taxes on inventory,
    - interest (paid on the capital invested in inventories),
    - the opportunity cost (e.g. the interest gained from money saved in the bank, the return from money invested in shares or bonds, or the yield gained by possible alternative use of funds),
    - cost due to the possibility of spoilage or obscolescence.

Holding and carrying cost can be expressed either in CZK per unit per time period (usually a year) or in percentage of the annual value of the inventory.

4. *Shortage (or stockout) cost* is significant if shortage of items affects the company's activities, revenue, profit, employment inside the company etc. In general, shortage leads to unsatisfied demand for the required items. Following examples of such cost can be considered:

   - cost of idled production, cost of idled machines (in case of the spare-parts storage),
   - cost of placing special expensive expediting order to restore the inventory,
   - loss of customers due to late deliveries of the finished goods,
   - other cost, often incalculable (the shortage of blood or ambulance in garage).

Consideration of these four cost items depends on the real specifications of the analyzed inventory system and thereby also on the used inventory model. Before providing a classification of inventory models and analyzing some of them, several additional terms are necessary to be explained [8].

Inventory Level

This is the available size of the inventory, i.e. the number of stocked items (the number of units) or the amount of stocked material (in kilograms, liters, meters, etc.).

Demand and Depletion

The purpose of making inventory corresponds to specific *demand*. In production process, for example, the production technology determines amount of items required within specific period. The *rate of demand* then determines the inventory *depletion rate* (and hence, the inventory level). The higher the rate of demand is, the quicker the inventory level is being reduced. The rate of demand can be constant throughout time or can fluctuate.

Reordering, Reorder Point, and Lead Time

Since the inventory is being depleted, stocked items must be replenished periodically. When the inventory level is reduced to a signal level called the *reorder point*, the *replenishment order* must be placed to restore the inventory on time. The time period between placing the order and receiving the shipment is called the *lead time*. This period can be constant or variable.

Shortages, Surpluses, and Safety Stock

In an ideal case, the shipment arrives at the moment when the last unit of inventory has been depleted. Wrong timing of real replenishment order may cause the future unexpected *shortage* of the inventory (if we make the order after the optimal reorder date) or *surplus* of units in the warehouse (if we make the order before the optimal reorder date). To prevent the shortage (its full elimination or partial reduction) company's management can keep a buffer in the form of *safety stock* that is used in case where shortage event would occur. This issue is the most significant in the models with probabilistic demand or lead time.

Average Inventory

Concept of an average inventory is used in almost all the inventory models. As a short illustration of this very important term of inventory management, we introduce the following example:

Example 4.1
Monitoring the inventory level during a five-day period, we get the following numbers:

Day	Units
Monday	200
Tuesday	150
Wednesday	100
Thursday	50
Friday	0

The average inventory level is then:

$$AI = \frac{200 + 150 + 100 + 50 + 0}{5} = 100 \text{ units.}$$

Figure 4.1 shows the depletion of the inventory in time (decrease of the inventory level occurs at the beginning of each day). Since demand is constant in time, the average inventory can be also computed as (200+0)/2 = 100 units.

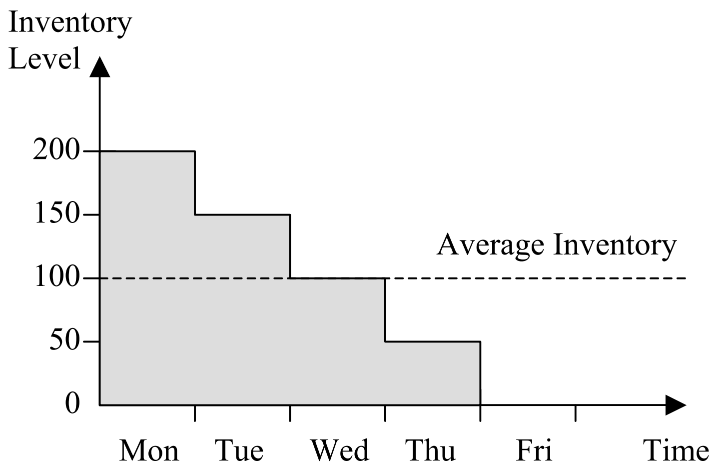


Fig. 4.1 Average Inventory

Classification of Inventory Models

As mentioned above, a type of demand determines the formulation of the problem and, consequently, the applied model. From this point of view, the inventory models can be divided into *deterministic models* and *probabilistic models*.

In **deterministic models** the rate of demand (and the inventory depletion rate) is known with certainty. As an example we can consider the production process with the constant production rate. This rate (e.g. number of units produced per day) determines the constant rate of demand for the inventory (e.g. semi-products) and hence, the uniform decrease in the inventory level. These are some of such models:

- Basic economic order quantity model (EOQ model).
- EOQ model with back orders allowed (with planned shortages).
- Economic production lot size model.
- EOQ model with quantity discounts.

In **probabilistic models** the rate of demand (and the inventory depletion rate) is probabilistic (described as a random variable). To illustrate such models let us consider a large supermarket with complex inventory system. Demand for any goods offered in the supermarket is obviously random. Demand level (in day, week or month) actually determines real depletion of the inventory. It is evident that monitoring and analyzing the demand fluctuation in time is a complex and never-ending process for a highly experienced team of managers and database analysts.

Two basic models will be described later:

- Probabilistic model with continuous demand.
- Single-period decision model.

Figure 4.2 illustrates that it is possible to extend the classification of the inventory models according to demand from another point of view [7].

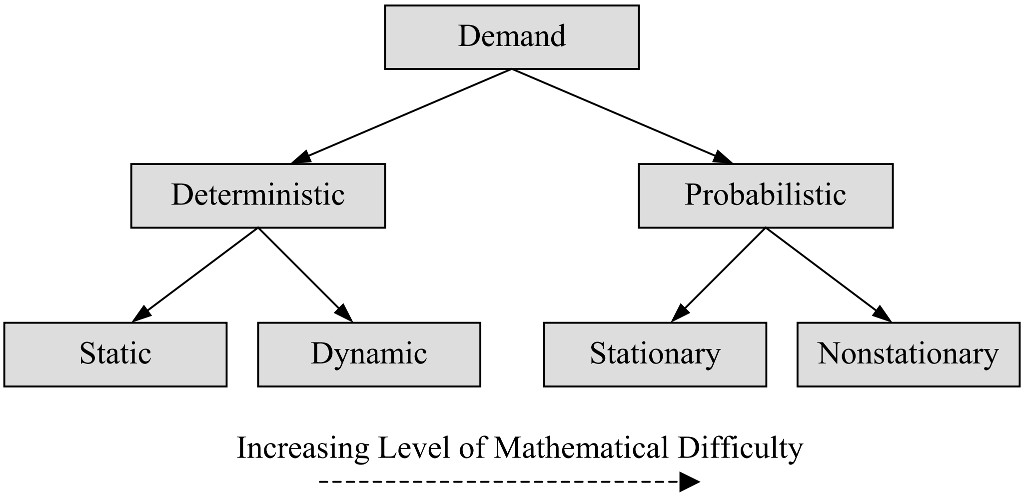


Fig. 4.2 Classification for Demand in Inventory Models

1. In case of a **static** demand, its rate is known with certainty (deterministic model) and in addition, it does not change from one time period to the next. This type of demand is considered in the simplest inventory models.

2. Although in case of a **dynamic** demand its rate is known with certainty, it is not constant throughout time. Thus, we know the demand value (or its rate) for each period.

3. A **stationary** probabilistic demand corresponds to a random variable with the probability density function, remaining unchanged over the time.

4. In a **nonstationary** case of demand, the corresponding probability density function varies within time period. Considering this type of demand leads to the most complex inventory models in which the simulation process is often used as the last chance of successful inventory managing.

4.2 Deterministic Inventory Models

As defined before, in the deterministic models we assume known rate of demand. Although the rate in such models may change throughout time (the change is known with certainty), we concentrate only on the *static deterministic models*. Thus, the *constant demand* (without any change from one time period to the next) is considered in all the models described in this section.

4.2.1 The Economic Order Quantity Model (EOQ)

EOQ model is the simplest of all the inventory models. Two basic questions associated with the inventory management (how much and when to order) can be extended to the following list of questions:

1. How much (optimum quantity) should be ordered?
2. When (in optimum reorder point) should the order be placed?
3. What is the total cost?
4. What is the average inventory level?
5. What is the maximum inventory level?

Because of the complexity of each real inventory system, we must introduce some assumptions for each constructed model.

Assumptions of EOQ model:
- single item is considered,
- demand for the item is known and constant over the time,
- lead time is known and constant over the time,
- uniform depletion of the inventory is supposed,
- order quantity is constant,
- purchasing cost is independent of the order quantity,
- unit holding cost is independent of the order quantity,
- replenishment is executed exactly on the point when the shipment arrives and the inventory level reaches zero (no shortages or surpluses are permitted).

Respecting all the assumptions we can graph the behavior of the EOQ model as it is shown in Figure 4.3. The ordering and replenishment processes are organized periodically in constant cycles (we consider three cycles) and all the order quantities are equal. Each cycle begins with the inventory replenishment and finishes with the total depletion of the store.

Considering the assumptions, the inventory level is being decreased with the uniform rate between the beginning and the end of the cycle. Thus, this process corresponds to the downward lines in Figure 4.3. Since each order arrives exactly on the point of the total depletion, no shortages occur in the inventory process.

Two important inventory levels are outlined in the graph: the maximum level and the average level. In the EOQ model the maximum level equals to the order quantity (denoted as q). Hence, the average level is $q/2$. Within each cycle, of course, a reorder point could be found (for better readability of the figure these points and corresponding lead times are not marked in the graph).

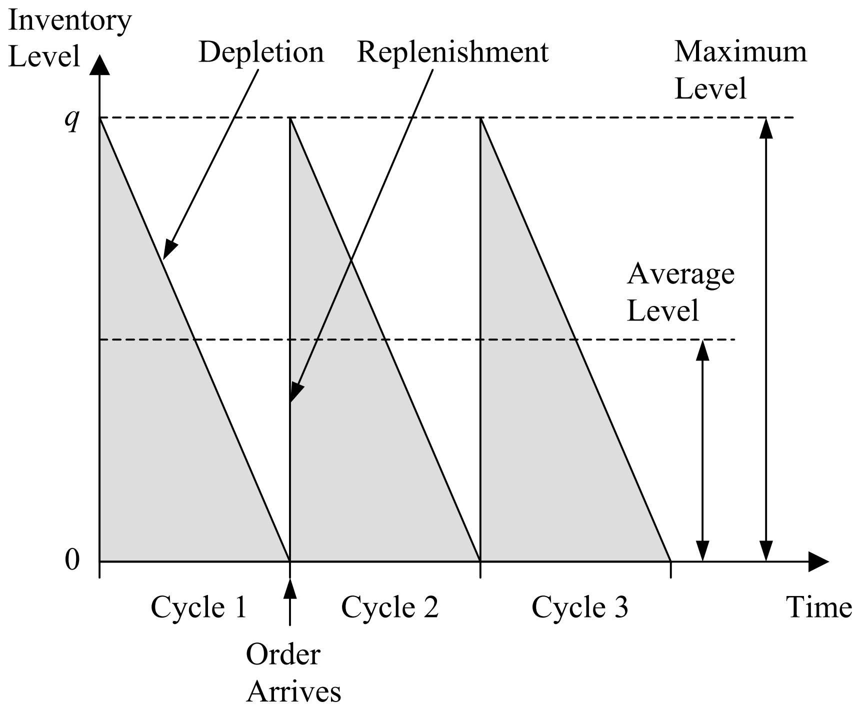


Fig. 4.3 Economic Order Quantity Model

Example 4.2

The private brewery produces monthly 4 000 hl of beer. 25% of the production is planned to be filled into glass bottles. The empty bottles are stored in plastic cases (each case contains 20 bottles) and the average annual holding cost per case is 20 CZK. The carrier, transporting the cases into the brewery's store, charges a fixed cost associated with each order for 11 000 CZK. In addition, the brewery's own fixed cost of 1 000 CZK per each order is necessary to be involved into the final calculation. The lead time between the placing each order and its delivery is 1/2 of month. Since filling of the bottles is the uniform process the inventory depletion rate is uniform as well.

Management of the brewery decided to analyze the inventory system in order to minimize the total cost associated with inventory replenishment and holding of the bottles in the store.

These questions should be answered:

1. What is the optimum quantity order?
2. What are the maximum level and the average level of inventory?
3. What is the minimum total annual cost?
4. When to order?
5. How many orders are placed per year?

Solution

1. Define input parameters & decision variables

In the example, following *parameters* of the model can be recognized:

Q – annual demand for cases,

c_1 – average annual holding cost per case,

c_2 – ordering cost per order,

d – lead time (in years) between placing an order and receiving delivery.

Referring to the managerial questions, we introduce the following *variables* into the model:

q – order quantity (number of cases in each order),

n – number of orders placed within a year,

OC – total annual ordering cost,

HC – total annual holding cost,

TC – total annual cost,

$q_{\max}$ – maximum inventory level,

q_{avg} – average inventory level,

t – time interval (in years) between two consecutive orders (length of the inventory cycle),

r – reorder point (level of inventory at reordering).

First, we must calculate (or recalculate) the parameters in order to make them comparable in terms of period (annual values) and unit of measurement (a case). All the variables, of course, must respect this notation.

(a) **Annual demand $\boldsymbol{Q}$**. The monthly production is 4 000 hl = 400 000 liters of beer. Only 25% of the production is to be filled into the bottles, i.e. 100 000 liters. Considering the half-liter bottles, the monthly demand corresponds to 200 000 bottles, i.e. 10 000 cases. Hence the annual demand $\boldsymbol{Q}$ **= 120 000 cases**.

(b) **Annual holding cost $\boldsymbol{c_1}$**. This parameter was specified (in the definition of the problem) as $\boldsymbol{c_1}$ **= 20 CZK per case**.

(c) **Ordering cost $\boldsymbol{c_2}$**. Since placing each order involves 11 000 CZK and 1 000 CZK, the calculated cost is $\boldsymbol{c_2}$ **= 12 000 CZK per order**.

(d) **Lead time $\boldsymbol{d}$**. As mentioned above, all considered time periods are calculated in years. Because the lead time is 1/2 of month, we get $\boldsymbol{d}$ **= 1/24 of year**.

2. Define the function of the total annual cost

The **total annual cost $\boldsymbol{TC}$**, including the total annual holding cost and the total annual ordering cost, can be expressed as follows:

$$\boxed{TC = HC + OC\,.} \tag{4.1}$$

Before analyzing this equation in detail, we define two other variables in the model. As it is shown in Figure 4.3 (following all assumptions of the EOQ model), the **average inventory level $\boldsymbol{q_{\text{avg}}}$** can be calculated as:

$$q_{avg} = \frac{q}{2} \tag{4.2}$$

and the **maximum inventory level q_{max}** equals to the order quantity:

$$q_{max} = q\,. \tag{4.3}$$

The **total annual holding cost *HC*** in Equation 4.1 is then given as the annual holding cost per case, multiplied by the average inventory level (average number of cases on hand):

$$HC = c_1 q_{avg} = c_1 \frac{q}{2}\,. \tag{4.4}$$

Since **the number of orders placed within a year** is:

$$n = \frac{Q}{q}, \tag{4.5}$$

the **total annual ordering cost *OC*** in (4.1) can be expressed as ordering cost per order, multiplied by the number of orders:

$$OC = c_2 n = c_2 \frac{Q}{q}\,. \tag{4.6}$$

Thus, introducing (4.4) and (4.6) into (4.1) the **total annual cost *TC*** is the function of the only variable q:

$$TC(q) = c_1 \frac{q}{2} + c_2 \frac{Q}{q}\,. \tag{4.7}$$

Figure 4.4 shows all the three total annual cost functions *HC*, *OC* and *TC*. Whereas *HC* is in direct proportion with q (a linear function), *OC* is in indirect proportion with q (a hyperbola). The function *TC* is then the graphical sum of these two curves. In the graph we marked three possible points (total annual cost) corresponding to different order quantities 10 000, 60 000 and 120 000 cases. Detailed cost calculation for three prospective policies can be found in Table 4.1.

	Policy I	Policy II	Policy III
Annual demand Q	120 000	120 000	120 000
Order quantity q	10 000	60 000	120 000
Annual holding cost per case c_1	20	20	20
Average inventory level $q_{avg} = q/2$	5 000	30 000	60 000
Total annual holding cost *HC*	**100 000**	**600 000**	**1 200 000**
Ordering cost per order c_2	12 000	12 000	12 000
Number of orders $n = Q/q$	12	2	1
Total annual ordering cost *OC*	**144 000**	**24 000**	**12 000**
Total annual cost *TC*	**244 000**	**624 000**	**1 212 000**

Tab. 4.1 Annual Inventory Cost Calculation

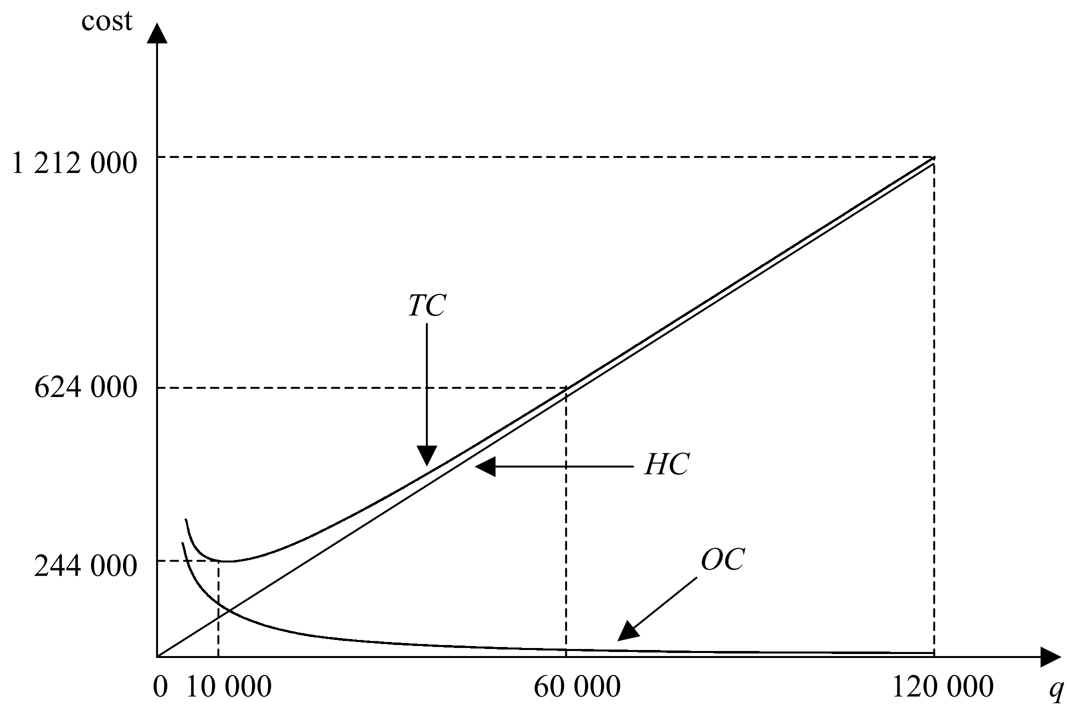


Fig. 4.4 Annual Inventory Cost

For better understanding of the delivery cycles and depletion of the inventory in proposed policies, see Figure 4.5 (time is measured in months). While Policy I seems to be quite reasonable, Policy III might be even irrealizable because of the brewery's store capacity. The lower ordering frequency is, the more of store space is occupied on the point of replenishment.

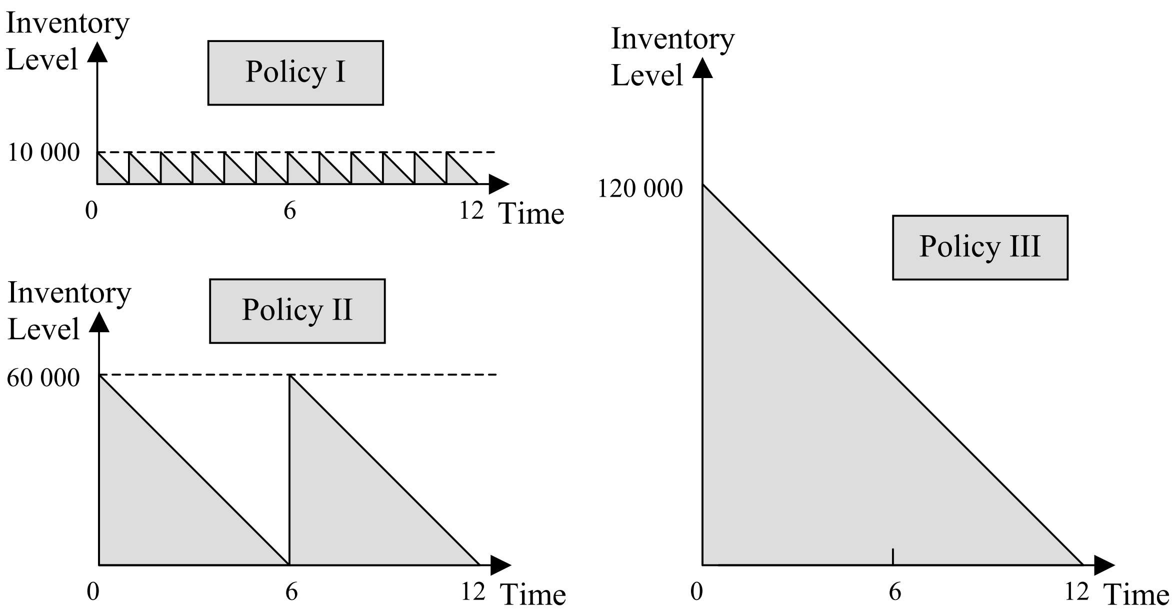


Fig. 4.5 Different Ordering Policies

3. The objective is to minimize the total annual cost

As it has been already shown (Equation 4.7), the total annual cost can be expressed as:

$$TC(q) = c_1 \frac{q}{2} + c_2 \frac{Q}{q}.$$

Minimizing this function, we will find the optimum order quantity. Since $TC(q)$ is a continuous function with the only one minimum point (see Figure 4.4), the optimum value of q is obtained by solving the following equation (of the only one unknown q):

$$\boxed{\frac{dTC(q)}{dq} = \frac{c_1}{2} - \frac{c_2 Q}{q^2} = 0.} \tag{4.8}$$

Thus, the **optimum order quantity** is:

$$\boxed{q^* = \sqrt{\frac{2Qc_2}{c_1}}.} \tag{4.9}$$

Note: to be mathematically precise, we should verify whether this value corresponds to the minimum point of the analyzed function. Since the second derivative of $TC(q)$ is positive, the point is the minimum point indeed.

Introducing the values of all the parameters (Q, c_1 and c_2) into Equation 4.9 we obtain the optimum order quantity in the example:

$$q^* = \sqrt{\frac{2(120\ 000)(12\ 000)}{20}} = 12\ 000 \text{ cases.}$$

If the brewery's management (considering the annual demand of 120 000 cases) wants to minimize the total annual cost associated with replenishment and holding bottles in the store, 10 orders of 12 000 cases should be placed per year (Equation 4.5).

Substitution of (4.9) into (4.7) leads, after the simplification, to the formula of the **optimum total annual cost**:

$$\boxed{TC^* = \sqrt{2Qc_1c_2}.} \tag{4.10}$$

Using this equation in the example, the optimum value of the total annual cost is:

$$TC^* = \sqrt{2(120\ 000)(20)(12\ 000)} = 240\ 000 \text{ CZK.}$$

Figure 4.6 apparently shows that the optimum value TC^* occurs at the intersection of HC and OC. Hence, an alternative approach to determination of the optimum order quantity is solving the following equation of unknown q:

$$c_1 \frac{q}{2} = c_2 \frac{Q}{q}.$$

By easy manipulation of this equation we get the formula (4.9).

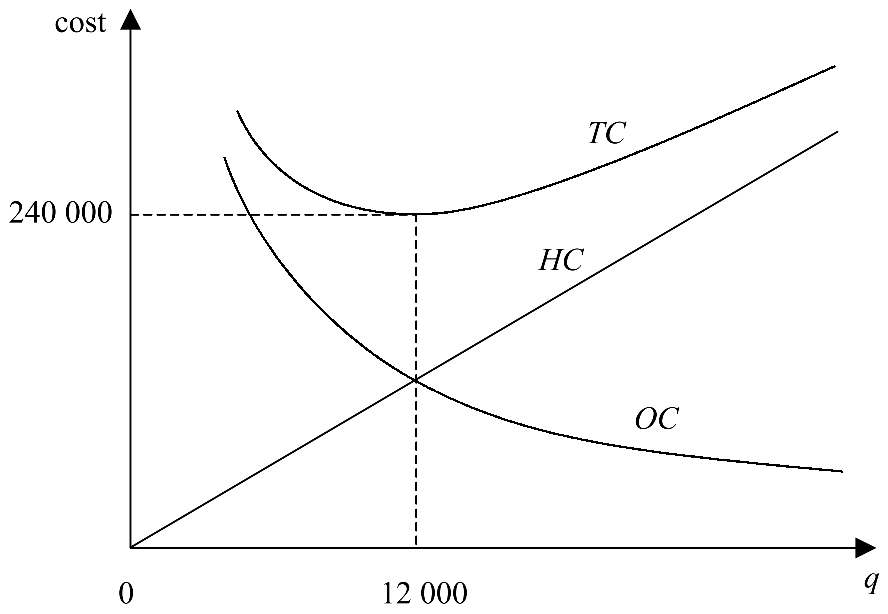


Fig. 4.6 Optimum Order Quantity & Total Annual Cost

The optimum ordering frequency n^* (number of orders placed within a year) determines the **optimum length of the inventory cycle** t^* - this is a reciprocal value of n^*:

$$\boxed{t^* = \frac{1}{n^*} = \frac{q^*}{Q} = \sqrt{\frac{2c_2}{Qc_1}}\,.} \tag{4.11}$$

Since in the example $n^* = 10$, the optimum length $t^* = 1/10$ of year. Supposing a continuous production process throughout a year, the management would place orders in the intervals of 365/10 = 36.5 days. In Figure 4.7 all 10 delivery cycles are graphed in real time (in days). See the maximum and the average inventory level in the graph.

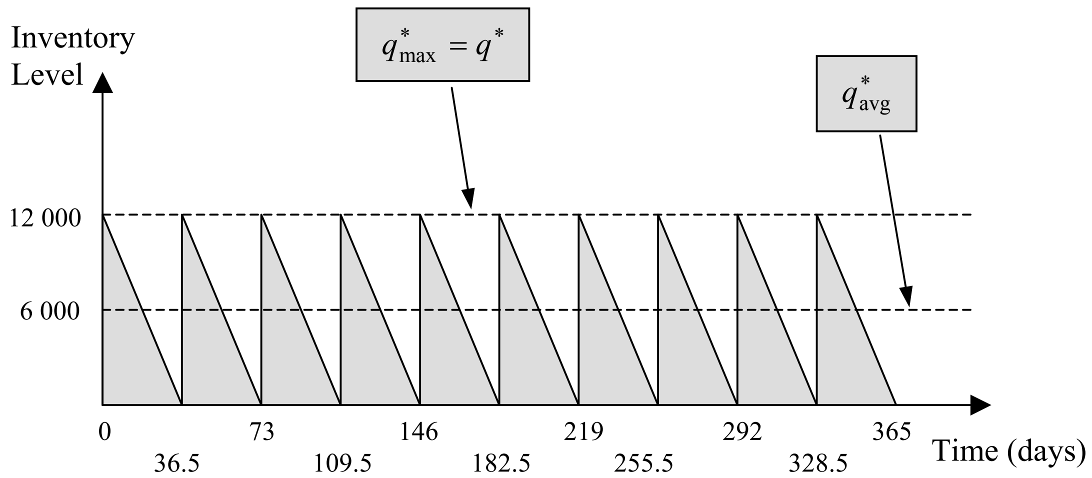


Fig 4.7 Optimum Delivery Cycles

The last management's decision concerns a reorder point. As defined before, this is the alarming inventory level that requires placing a new order. This value depends, of course, on the lead time between placing order and delivery. Figure 4.8 shows the relations between all the values necessary for determination of the optimum reorder point r^*. Using the elementary mathematics (the parallel triangles theorem) we can write:

$$\frac{r^*}{d} = \frac{q^*}{t^*}.$$

Hence, the **optimum reorder point r^*** is:

$$\boxed{r^* = \frac{dq^*}{t^*} = dQ.} \tag{4.12}$$

Thus, in the example, we get:

$$r^* = \frac{1}{24} 120\,000 = 5\,000 \text{ cases.}$$

If the inventory level decreases to 5 000 cases, the management must place a new order to prevent discontinuity of the production process due to cases shortage in the future.

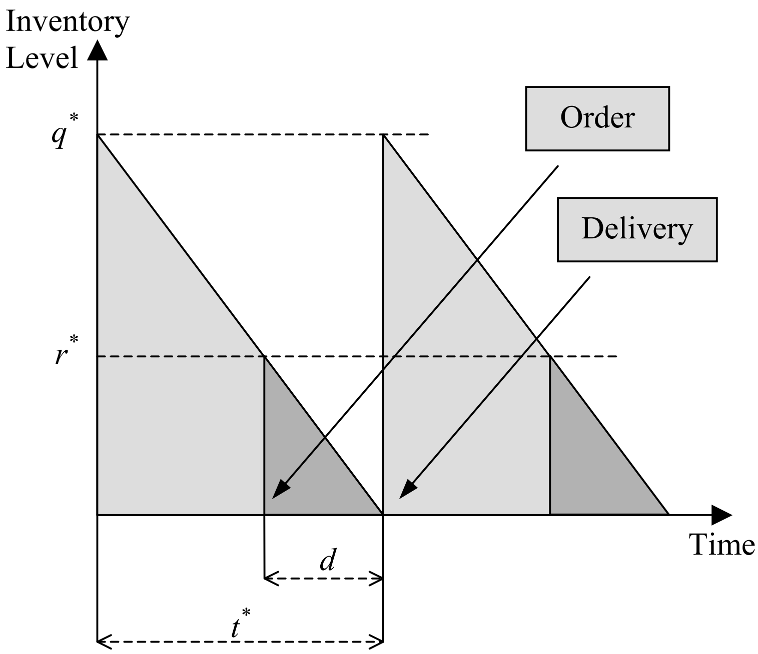


Fig 4.8 Reorder Point

Equation 4.12 is valid only in case of $d \le t^*$. If $d = t^*$, a new order should be placed exactly in the moment when the previous order is delivered. However, if $d > t^*$, formula (4.12) is inconsistent with a logical assumption $r^* \le q^*$, and should be modified as follows:

$$\boxed{r^* = \frac{dq^*}{t^*} \mod q^* = dQ \mod q^*.} \tag{4.13}$$

The operator “modulo” returns the remainder for division of the first argument by the second. Suppose d = 1/6 instead of 1/24 (the lead time is 2 months instead of 1/2 of month). The calculation of (4.13) gives r^* = 20 000 mod 12 000 = 8 000 cases.

4. Summary

This is the summary for the brewery's management to make the optimal decisions in the company's inventory process.

Each year **10 orders** will be placed. The order quantity is **12 000 cases**. The interval between placing two consecutive orders is **36.5 days**. This value represents the length of delivery cycle. When the inventory in the store decreases to **5 000 cases** a new order must be placed.

Following this optimal policy, the brewery's total annual cost associated with the inventory of cases will be **240 000 CZK**.

The average inventory level is **6 000 cases**, the maximum level is **12 000 cases**.

Note 1: As mentioned above, the available space in the store might be a significant restriction of the model in case that the maximum (and optimum) inventory level would be higher than (or very close to) the store capacity. Mostly, in the real process it is not possible to implement exactly 36.5 days between two consecutive orders, but this is not a major problem. More important issue seems to be the minute of replenishment. In a real company the replenishment is being started before the total depletion of the inventory. Thus, in the real life the maximum inventory level might be rather higher than the computed value to keep the continuity of the production process.

Note 2: The computed order quantity (or other results as e.g. the number of ordering cycles) in the EOQ model may be noninteger. After rounding the result both down and up we calculate the total cost and then select the better alternative.

4.2.2 The EOQ Model with Planned Shortages

In the elementary EOQ model we assume that the store is replenished exactly on the point when the inventory level reaches zero. Thus, no shortage occurs in such situation. However, in many situations it is possible to allow shortages, or even to plan them rather then to build a safety stock. Since each shortage affects the P&L (profit and loss) of a company, specific cost associated with the stockout must be considered in the inventory management. Such cost may be a loss of the profit, or an extra charge that must be paid if, for example, other activities (e.g. continuous production process) depend on the inventory.

Example 4.3

We will modify Example 4.2, in which the brewery's inventory process was analyzed. We allow the shortage of cases in the company's store. Since the production (filling bottles) is continuous process that cannot be interrupted, in case of stockout it is possible (in a short term) to use a small flexible store with a safety stock. Cases from this house can be supplied into the production process directly and in a uniform flow (similarly to the large store). Using this possibility, the company must pay extra annual charge (150 CZK per case) corresponding to the average annual operating cost on the inventory in the small store (no holding host is considered). For simplicity we will call the large store as the *primary store* and the smaller store as the *secondary store*.

The brewery's management should make decisions on the order quantity and the period in which reorders would be placed to minimize the total cost.

Solution

1. Define input parameters & decision variables

As Figure 4.9 shows, the considered inventory process is quite different from the previous example.

Some of the parameters and variables, considered in the elementary EOQ model, must be re-defined. In addition, some new parameters and variables must be introduced.

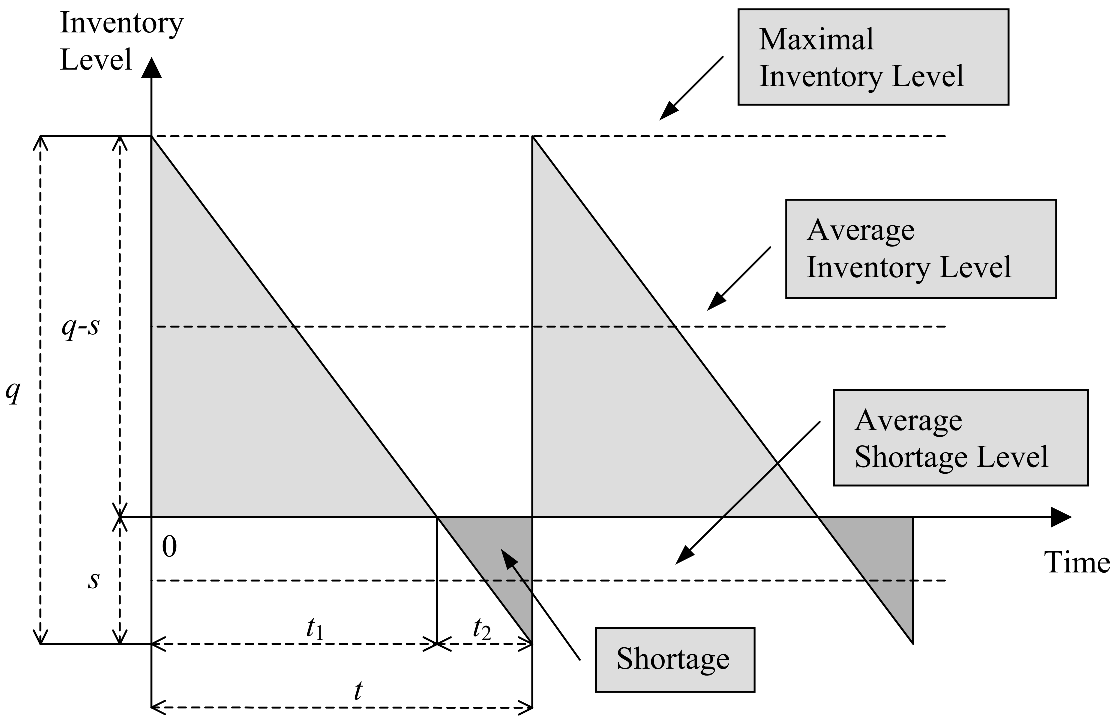


Fig. 4.9 EOQ Model with Planned Shortage

The following *parameters* are defined in the brewery's model with planned shortages:

Q – annual demand for cases,

c_1 – average annual holding cost per case (in the primary store),

c_2 – ordering cost per order,

c_3 – average annual operating cost per case (in the secondary store),

d – lead time (in years) between placing an order and receiving delivery.

The list of *variables* follows:

q – order quantity (number of cases in each order),

s – shortage level (inventory level in the secondary store),

n – number of orders placed within a year,

OC – total annual ordering cost,

HC – total annual holding cost (concerning with the primary store),

SC – total annual shortage cost (operating cost associated with the secondary store)

TC – total annual cost,

$q_{\max}$ – maximum inventory level,

q_{avg} – average inventory level,

$s_{\max}$ – maximum shortage level (maximum inventory level in the secondary store),

s_{avg} – average shortage level (average inventory level in the secondary store),

t – time interval (in years) between two consecutive orders (length of the inventory cycle),

t_1 – time period (in years) during which no stockout occurs in the primary store,

t_2 – time period (in years) during which the inventory in the secondary store is used,

r – reorder point (level of inventory at reordering).

2. Define the function of the total annual cost

Referring to Figure 4.9, we will carry out the analysis of one inventory cycle.

The **average inventory level q_{avg}** in the cycle can be expressed as:

$$\boxed{q_{avg} = \frac{q-s}{2}} \tag{4.14}$$

and the **average shortage level s_{avg}** (the average inventory level in the secondary store):

$$\boxed{s_{avg} = \frac{s}{2}\,.} \tag{4.15}$$

The **maximum inventory level $q_{\max}$** equals to the order quantity minus the shortage level:

$$\boxed{q_{\max} = q - s\,.} \tag{4.16}$$

The **maximum shortage level $s_{\max}$** corresponds, in the example, to the maximum inventory level in the secondary store.

Since the holding cost in the primary store relates to the period t_1 (only in this period the store is not empty), the **total holding cost *HC*** in one cycle can be calculated as follows:

$$\boxed{HC = c_1\, q_{avg}\, t_1 = c_1 \frac{q}{2} t_1\,.} \tag{4.17}$$

Similarly, the **total shortage cost *SC*** in one cycle corresponding to the period t_2 (only in this period the secondary store is used) is:

$$\boxed{SC = c_3\, s_{avg}\, t_2 = c_3 \frac{s}{2} t_2 .} \tag{4.18}$$

Note: In the beginning of the period t_1 the secondary store is logically replenished (the inventory level is $s_{\max}$) and then left waiting (consider that there is no holding cost) all this period without any use (the door is being locked). After the primary store is totally depleted the door of the secondary store is open and the inventory starts to be used with the only operating cost. Within the period t_2 the inventory in this store is depleted as well and replenished again (together with the primary store) at the end of the inventory cycle.

Analyzing the **total ordering cost *OC*** in one cycle, we should realize that each cycle is actually determined by one order. Thus, we can simply write:

$$\boxed{OC = c_2 .} \tag{4.19}$$

Total cost associated with one cycle (and with one order) can be calculated as $HC + SC + OC$.

If we consider that **the number of orders placed within a year** is:

$$\boxed{n = \frac{Q}{q},} \tag{4.20}$$

the **total annual cost *TC*** can be then expressed as:

$$TC = (HC + SC + OC)\, n ,$$

or as the function of two variables q and s:

$$\boxed{TC(q,s) = (c_1 \frac{q-s}{2} t_1 + c_3 \frac{s}{2} t_2 + c_2) \frac{Q}{q} .} \tag{4.21}$$

The variables t_1 and t_2 in Equation 4.21 can be expressed (using the parallel triangles theorem and considering $t = 1/\mathrm{n} = q/Q$) as follows:

$$t_1 = \frac{q-s}{Q},$$

$$t_2 = \frac{s}{Q} .$$

Introducing these formulas into (4.21) we get (after the simplification):

$$TC(q,s) = c_1 \frac{(q-s)^2}{2q} + c_3 \frac{s^2}{2q} + c_2 \frac{Q}{q} . \tag{4.22}$$

3. The objective is to minimize the total annual cost

The function (4.22) is actually the function of the only two variables q and s. In order to find the minimum of TC, its partial derivatives with respect to q and s are set to zero.

Solving two generated equations simultaneously, we get as the results the **optimum order quantity**:

$$\boxed{q^* = \sqrt{\frac{2Qc_2}{c_1}}\sqrt{\frac{c_1 + c_3}{c_3}}} \tag{4.23}$$

and the **optimum (maximum) shortage level**:

$$\boxed{s^* = s_{\max} = q^* \frac{c_1}{c_1 + c_3}\,.} \tag{4.24}$$

In the example, the optimum order quantity is:

$$q^* = \sqrt{\frac{2(120\ 000)(12\ 000)}{20}}\sqrt{\frac{20+150}{150}} \doteq 12\ 774.98 \text{ cases}$$

and the optimum shortage level:

$$s^* \doteq 12\ 774.98 \frac{20}{20+150} \doteq 1502.94 \text{ cases}\,.$$

The optimum number of orders placed in the year is:

$$n^* = \frac{Q}{q^*} \doteq 9.39$$

and the optimum length of the delivery cycle:

$$t^* = \frac{q^*}{Q} \doteq 0.11 \text{ years} \doteq 38.86 \text{ days}\,.$$

These noninteger values come into conflict with practical recommendations how often and how much to order. The successful manager cannot direct the inventory department to place orders of 12 774.98 cases in intervals of 38.86 days. To make reasonable decisions the manager should adjust the results. Whatever corrections will be made we lose the optimum solution. However, we can suggest such strategy that will be as close as possible to the optimal (but impracticable) strategy.

4. Correction of noninteger results

1. First, we round the optimum length of the delivery cycle. Two possibilities (rounding down or up) are considered:

$$t_A = 38 \text{ days}\,,$$

$$t_B = 39 \text{ days}\,.$$

2. Following the first integer alternative (strategy A), the company should make an order of "exactly" this quantity:

$$q_A^0 = 120\ 000 \frac{38}{365} \doteq 12493.15 \text{ cases}\,.$$

An alternative method how to calculate this value is based on the parallel triangle theorem (Figure 4.10):

$$\frac{q_A^0}{q^*} = \frac{t_A}{t^*} \quad \Rightarrow \quad q_A^0 = q^* \frac{t_A}{t^*} . \tag{4.25}$$

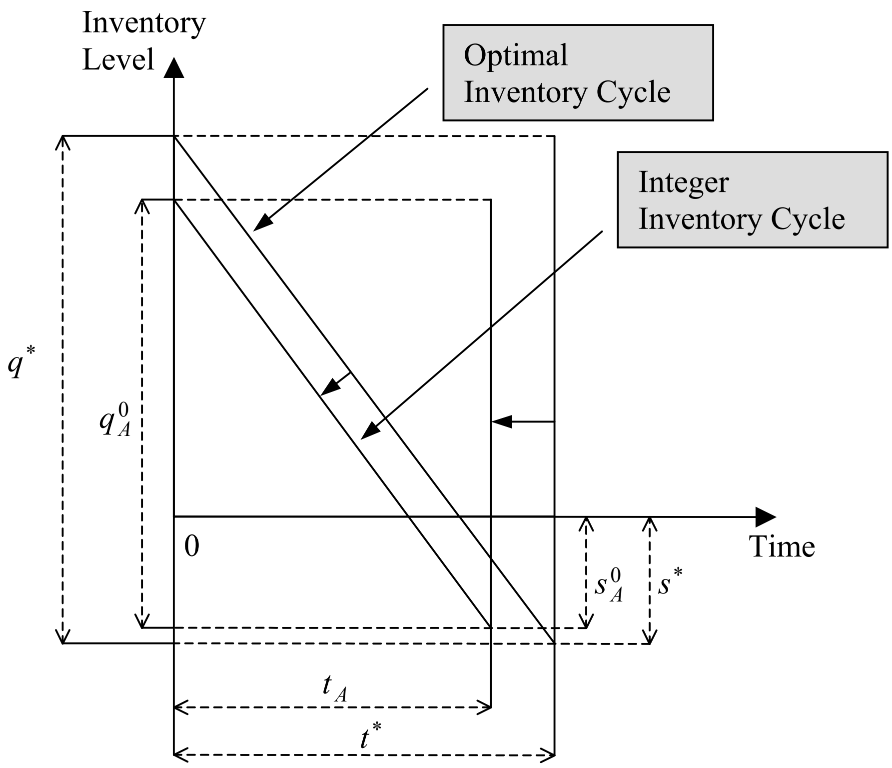


Fig. 4.10 Optimum and Integer Lengths of Inventory Cycle

As the value q_A^0 is noninteger, the management should round it down or up. Rounding down leads to infeasible solution: 12 493 (365/38) < 120 000. Rounding up to 12 494 cases satisfies (and exceeds) the annual demand (surplus is almost 9 cases).

Thus, the order quantities for both the strategies are computed and rounded up:

$$q_A = 12\,494 \text{ cases},$$
$$q_B = 12\,822 \text{ cases.}$$

3. Similarly to Equation 4.25, we can simply determine the maximum shortage level corresponding to 38 days (39 days):

$$\frac{s_A^0}{s^*} = \frac{q_A^0}{q^*} \quad \Rightarrow \quad s_A^0 = s^* \frac{q_A^0}{q^*} . \tag{4.26}$$

$$\frac{s_B^0}{s^*} = \frac{q_B^0}{q^*} \quad \Rightarrow \quad s_B^0 = s^* \frac{q_B^0}{q^*} . \tag{4.27}$$

After substitution of the exact values into (4.26) and (4.27) we get noninteger values:

$$s_A^0 \doteq 1\,469.78 \text{ cases},$$
$$s_B^0 \doteq 1\,508.46 \text{ cases}.$$

As in the brewery's inventory process these numbers are considered to be integer as well, the management has two possibilities for each strategy. Strategy A is branched into strategies $A1$ and $A2$, strategy B into strategies $B1$ and $B2$ as follows:

$$s_{A1} = 1\,469 \text{ cases}, \quad s_{A2} = 1\,470 \text{ cases},$$
$$s_{B1} = 1\,508 \text{ cases}, \quad s_{B2} = 1\,509 \text{ cases}.$$

Further in the text we are going to combine the integer order quantities with the integer shortage levels to compute the total cost and to make the best real decision.

4. Since the optimum length of the inventory cycle has been rounded (down or up), the used strategy causes that the last cycle in the current year is not the full cycle. Hence, the delivery will not come exactly at the end of the year and some cases remain either in the primary or in the secondary store.

Before cost calculations we compute the number of the inventory cycles:

$$n_A^0 = \frac{365}{38} \doteq 9.61,$$
$$n_B^0 = \frac{365}{39} \doteq 9.36.$$

Rounding down these values we get the number of full cycles:

$$n_A = n_B = 9.$$

5. Now we can project total annual cost as follows:

$$TC = TC^F \frac{365}{nt}, \tag{4.28}$$

where
- TC … total annual cost,
- TC^F … total cost relating to all the full periods,
- n … number of the full periods,
- t … length of the full period.

For calculation of TC^F we use Equation 4.22 (however, we must take $Q = nq$ instead of $Q = 120\,000$). Of course, we should consider all four strategies:

Strategy	t	q	n	$Q = nq$	s	$q_{\max} = q - s$	TC^F	TC
A1	38	12 494	9	112 446	1 469	11 025	218 241	232 918
A2	38	12 494	9	112 446	1 470	11 024	218 241	232 918
B1	39	12 822	9	115 398	1 508	11 314	221 135	229 956
B2	39	12 822	9	115 398	1 509	11 313	221 135	229 956

Tab. 4.2 Total Cost Estimation

In Table 4.2 the exact values of TC^F and TC are rounded to integers. Whereas differences between total cost for strategies *A1* and *A2* (or *B1* and *B2*) are not quite significant, crosswise differences could be considerable for the company.

Since the management's objective is to minimize the total annual cost, strategy B (either $B1$ or $B2$) should be used in the company's inventory process. Total cost is estimated in the amount of approximately 229 956 CZK.

For the calculation of optimum total annual cost we use Equation 4.22, in which the optimum (noninteger) order quantity and optimum (noninteger) shortage level are substituted:

$$TC(q^*, s^*) = c_1 \frac{(q^* - s^*)^2}{2q^*} + c_3 \frac{s^{*2}}{2q^*} + c_2 \frac{Q}{q^*} \doteq 225\,441 \text{ CZK} .$$

An application of the practicable "integer" strategy B increases total annual cost in about 2% (in comparison with the impracticable optimum total annual cost).

The maximum inventory levels $q_{\max}$ in the primary store for each strategy can be found in Table 4.2.

6. The last decision concerns with the reorder point. For the EOQ model with planned shortages we must change the elementary EOQ model's formula (4.13) as follows:

$$\boxed{r = \left(\frac{dq_B}{t_B} \mod q_B \right) - s_B .} \tag{4.29}$$

If we decide to follow the strategy $B1$, the reorder point is:

$$r = \left(\frac{\frac{1}{24} 12\,822}{\frac{39}{365}} \mod 12\,822 \right) - 1508 = 5\,000 - 1508 = 3\,492 \text{ cases.}$$

5. Summary

The brewery's management should place orders periodically in the intervals of **39 days**. This value also corresponds to the length of delivery cycle. The order quantity is **12 822 cases**. The maximum inventory level in the primary store is **11 314 cases**. When the inventory level decreases to **3 492 cases**, a new order must be placed. On the point of delivery, **11 314 cases** are stored in the primary store and **1 508 cases** are stored in the secondary store.

Following this strategy, the brewery's total annual cost associated with the inventory of cases is approximately **225 441 CZK**.

Note: Comparison of the total annual cost in case the shortages are allowable (**225 441 CZK**), with the cost in situation where no shortage may occur (**240 000 CZK** in Example 4.2) shows a possibility for the company's management to improve its decisions. Finding a reasonable solution (building and using the secondary store) leads to improvement of the objective (decreasing total cost).

4.2.3 The EOQ Model with Quantity Discounts

In real markets suppliers usually offer to customers *quantity discounts* as an incentive for large order quantities. The larger order quantity is, the lower purchase price per item a customer must pay. Unit holding cost is often derived from the purchase price, mostly as its percentage. Hence, larger lots lead to lower unit holding cost (the elementary EOQ model's assumption of fixed unit holding cost is not valid any more). In practice, quantity discounts are scheduled in several purchase price intervals.

Example 4.4

Suppose now that the brewery in Example 4.2 takes the cases from a supplier, offering quantity discounts scheduled in Table 4.3. Annual unit holding cost is calculated as 50% of unit purchase price.

Discount Category	Order Size [cases]	Purchase Price [CZK per case]	Unit Holding Cost [CZK per case]
1	0 to 4 999	46	23
2	5 000 to 14 999	40	20
3	15 000 and over	36	18

Tab. 4.3 Quantity Discounts Schedule

The brewery's management should again decide for the order quantity that would lead to the minimum total annual cost. No shortages are allowed.

Solution

1. Define input parameters & decision variables

Definitions of all the parameters and variables can be taken from the elementary EOQ model in Example 4.2. In addition the *parameter* c_q and the *variable PC* are considered in the model:

c_q – unit purchase price (per case) – for the discount category corresponding to the order quantity q,

PC – total annual purchase cost.

2. Define the function of the total annual cost

The **total annual cost *TC***, including the total annual holding cost, the total ordering annual cost and total purchase cost, can be expressed as follows:

$$\boxed{TC = HC + OC + PC\,.} \tag{4.30}$$

Since the total purchase cost PC is calculated as the unit purchase price, multiplied by the annual demand, the total annual cost TC can be expressed as the function of the variable q:

$$\boxed{TC(q) = c_1 \frac{q}{2} + c_2 \frac{Q}{q} + c_q Q\,.} \tag{4.31}$$

3. The objective is to minimize the total annual cost

Similarly to the elementary EOQ model, for each discount category we compute the **optimum order quantity**:

$$\boxed{q^* = \sqrt{\frac{2Qc_2}{c_1}}\,.} \tag{4.32}$$

Table 4.4 shows the computed EOQ using the various unit holding cost c_1 associated with three discount categories.

1. In the first category, the optimum order quantity is $q_1^* \doteq 11\,191$ (rounded up), what is not a relevant value, because the upper bound for this category is 4 999 cases. Thus, for the calculation of the total cost we take $q_1 = 4\,999$ cases instead of 11 191 cases.

2. The optimum order quantity for the second category $q_2^* = 12\,000$ (an exact value) falls within the interval <5 000; 14 999> and we can take $q_2 = 12\,000$ for the calculation of minimum total cost.

3. In the last category, the optimum order quantity $q_3^* = 12\,650$ (rounded up) is lower than the lowest possible quantity (15 000 cases) to qualify for a discount. Therefore we take $q_3 = 15\,000$.

Discount Category	Order Size	q^*	q	TC
1	0 to 4 999	11 191	4 999	5 865 546
2	5 000 to 14 999	12 000	12 000	5 040 000
3	15 000 and over	12 650	15 000	4 551 000

Tab. 4.4 Total Cost for Quantity Discounts

The total annual cost for each discount category is calculated using the formula (4.31):

$$TC(q_1) = 23\frac{4\,999}{2} + 12\,000\frac{120\,000}{4\,999} + 46(120\,000) = 5\,865\,546\,\text{CZK},$$

$$TC(q_2) = 20\frac{12\,000}{2} + 12\,000\frac{120\,000}{12\,000} + 40(120\,000) = 5\,040\,000\,\text{CZK},$$

$$TC(q_3) = 18\frac{15\,000}{2} + 12\,000\frac{120\,000}{15\,000} + 36(120\,000) = 4\,551\,000\,\text{CZK}.$$

4. Summary

As the objective is to minimize the total annual cost, the management should use the third discount category. The optimum order quantity is **15 000 cases**. Qualifying for the offered discount the purchase price is **36 CZK per case**. The total annual cost is **4 551 000 CZK**.

All the inventory characteristics (using formulas for the elementary EOQ model) could be, of course, determined.

4.2.4 The Economic Production Lot Size Model

The last deterministic model, presented in this textbook, concerns the situations where the inventory is not replenished through the single batch, but gradually over a time period. As there are many variations of those models, we describe the best known of them. The inventory cycle in this model consists of two phases: *production phase* and *nonproduction phase* (Figure 4.11).

In the production phase (within a period t_1), the store is uniformly replenished from the production batch. Simultaneously, the inventory is being depleted in this phase. In Figure 4.11 a gradual inventory buildup is obvious – it is produced more than it is demanded (a production rate is greater than a demand rate). When the peek of the inventory level is reached, the production run is stopped and the inventory is depleted only (within a period t_2). In the point of the total depletion of the inventory a new production batch is started and the inventory cycle is repeated. No shortages are allowed in this model.

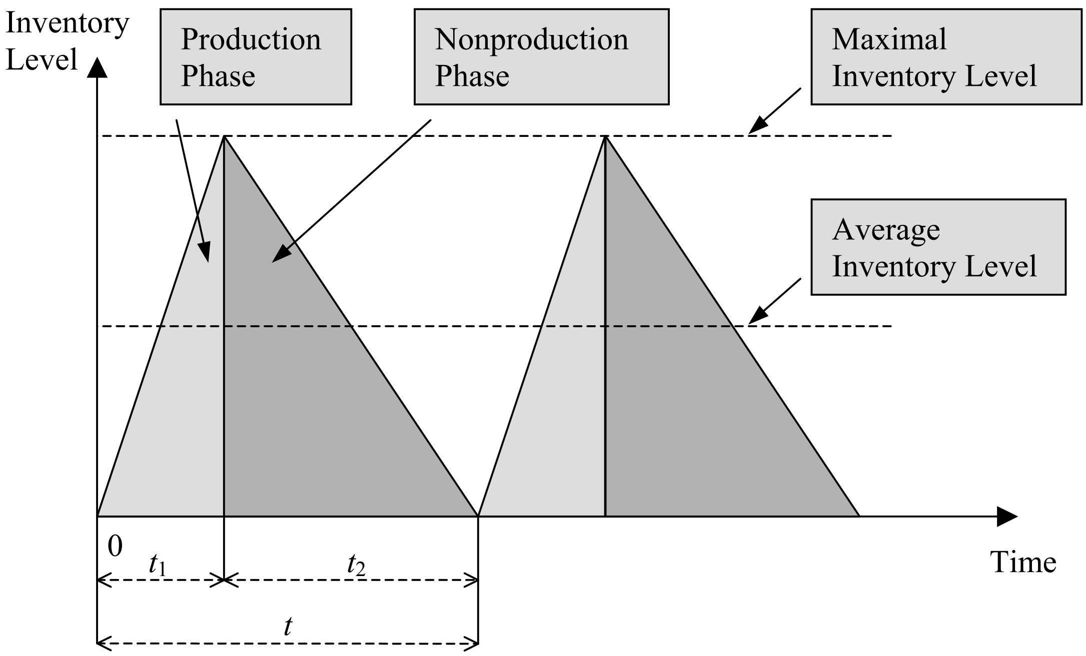


Fig. 4.11 Economic Production Lot Size Model

In the model we consider two cost items: *holding cost* and *production setup cost*. The definition of the holding cost is the same as in the EOQ model. Production setup cost is a fixed cost connected with a preparation of the production batch. This cost is independent of the production lot size, similarly, as the ordering cost is independent of the order quantity in the EOQ model.

The objective is to determine such production lot size q (economic lot size) that minimizes the total cost. It is evident that the maximum inventory level is always lower than the production lot size, and the average inventory level is one-half of the maximum inventory level.

Example 4.5

Modification of Example 4.2 consists now in the process of preparing the bottles for their filling. Instead of their simple ordering, the bottles are processed on the cleaning line. From this line the bottles are being transported (in cases) into the store and then from the store (when they are needed) directly to the filling process. We are not interested, in the example, in the processes preceding to the cleaning bottles (how and when the bottles are ordered, or where they are stored, etc.). However, we know the calculation of the cost connected with a preparation of the cleaning process (e.g. cost on servicing, setting up and washing the cleaning machine and the whole line). This cost was calculated to 12 000 CZK per one successive cleaning process. Suppose the original unit holding cost (annually 20 CZK per case) and the annual demand (120 000 cases). The cleaning line's daily output is 8 000 bottles. Preparation of the cleaning line takes 1/2 of month.

The brewery's management wants to determine the size of cleaning batch to minimize the total annual cost. Following questions should be answered to schedule production runs:

1. What is the optimum lot size?
2. What is the maximum level of the inventory?
3. What is the minimum total annual cost?
4. How long time does a cleaning run take?
5. When to start the preparation process (setup) for the cleaning run?

Solution

Although the brewery does not produce the bottles (or cases of bottles) - it just cleans them - we will call the process the production, since the models like this are actually used for production-consumption processes.

1. Define input parameters & decision variables

We consider following *parameters*:

Q – annual demand for cases,

c_1 – average annual holding cost per case,

c_2 – fixed setup cost per cleaning lot,

p – production rate (number of cleaned cases, e.g. per year),

h – demand rate (number of demanded cases, e.g. per year),

d – lead time (in years), necessary to prepare a new production run,

and these *variables*:

q – production lot size (number of cleaned cases in a cleaning lot),

n – number of lots within a year,

SC – total annual setup cost,

HC – total annual holding cost,

TC – total annual cost,

q_{max} – maximum inventory level,

q_{avg} – average inventory level,

t_1 – length of a production period (in years), when the production runs,

t_2 – length of a depletion period (in years), when only depletion proceeds,

t – cycle - time period (in years) between starts of two consecutive cleaning lots,

r – starting setup point (level of the inventory when the setup of the cleaning line should be started).

First, we must recalculate the parameters:

(a) **Production rate *p***. This parameter was specified as 8 000 cleaned bottles per day. Since our time unit is one year and the quantity unit is one case, we get 8 000*365/20 cleaned cases per year (if the cleaning line runs without any interruption). Thus ***p* = 146 000 cases per year**.

(b) **Lead time *d***. The lead time is 1/2 of month, i.e. ***d* = 1/24 of year**.

Note: In the example, if considered time unit is one year, the demand rate h equals to the annual demand Q. Generally, whatever time units of parameters p and h are used, they must be identical (mostly one day).

2. Define the function of the total annual cost

The **total annual cost *TC*** consists of the total annual holding cost and the total annual setup cost:

$$\boxed{TC = HC + SC\ .} \tag{4.33}$$

In the time period t_1 the total number of cleaned cases is pt_1. Since the demanded number of cases is ht_1, after this period the *maximum inventory level* is $pt_1 - ht_1 = (p - h)\, t_1$. Considering $pt_1 = q$ (i.e. $t_1 = q/p$), the **maximum inventory level q_{max}** can be expressed as follows:

$$\boxed{q_{max} = \frac{p-h}{p}\, q\ ,} \tag{4.34}$$

and the **average inventory level q_{avg}** can be calculated as follows:

$$\boxed{q_{avg} = \frac{q_{max}}{2} = \frac{p-h}{p}\frac{q}{2}\ .} \tag{4.35}$$

The **total annual holding cost *HC*** is defined as the annual holding cost per case, multiplied by the average inventory level:

$$\boxed{HC = c_1 q_{avg} = c_1 \frac{p-h}{p}\frac{q}{2}\ .} \tag{4.36}$$

Since the **number of production lots within a year** is:

$$\boxed{n = \frac{Q}{q}\ ,} \tag{4.37}$$

the **total annual setup cost *SC*** in (4.33) can be expressed as setup cost per cleaning lot, multiplied by the number of lots:

$$SC = c_2 n = c_2 \frac{Q}{q}. \tag{4.38}$$

Thus, substituting (4.36) and (4.38) into (4.33), the **total annual cost *TC*** is the function of the variable q:

$$TC(q) = c_1 \frac{p-h}{p} \frac{q}{2} + c_2 \frac{Q}{q}. \tag{4.39}$$

3. The objective is to minimize the total annual cost

After setting the first derivative of (4.39) to zero we get the **optimum (economic) lot size**:

$$q^* = \sqrt{\frac{2Qc_2}{c_1}} \sqrt{\frac{p}{p-h}}. \tag{4.40}$$

Introducing the values of all the parameters into Equation 4.40 we obtain the optimum lot size in the brewery:

$$q^* = \sqrt{\frac{2(120\,000)(12\,000)}{20}} \sqrt{\frac{146\,000}{146\,000 - 120\,000}} \doteq 28\,436.16 \text{ cases.}$$

Substituting the value of q^* into (4.34), the **maximum inventory level** is $q^*_{\max} \doteq 5\,064$ cases.

Substitution of (4.40) into (4.39) leads, after the simplification, to the formula of the **optimum total annual cost**:

$$TC^* = \sqrt{2Qc_1c_2} \sqrt{\frac{p-h}{p}}. \tag{4.41}$$

Using this equation in the example, the optimum value of the total annual cost is:

$$TC^* = \sqrt{2(120\,000)(20)(12\,000)} \sqrt{\frac{146\,000 - 120\,000}{146\,000}} \doteq 101\,279.49 \text{ CZK.}$$

Since $pt_1 = q$, the **optimum length of the production period** can be expressed as follows:

$$t_1^* = \frac{q^*}{p}. \tag{4.42}$$

Thus, in the example, $t_1^* = 0.1948$ years (that corresponds to 71.1 days).

Similarly, because of $ht_2 = q_{max}$, the **optimum length of the depletion period** is:

$$t_2^* = \frac{q_{\max}^*}{h} = \frac{p-h}{ph} q^* . \tag{4.43}$$

The value of this period, in the example, is $t_2^* \doteq 0.0422$ years (i.e. 15.4 days).

Since $t = t_1 + t_2$, the **optimum cycle** (time between starts of two consecutive cleaning lots) is $t^* \doteq 0.1948 + 0.0422 = 0.237$ years (86.5 days). For determination of this value, of course, the well-known formula can be used:

$$t^* = \frac{q^*}{Q} . \tag{4.44}$$

The last question concerns with the **optimum starting setup point**. Similarly to the EOQ model (where this point is called reorder point), it is an alarm inventory level to start preparations for a new cleaning run (a new order in the EOQ model). The way of calculation of the starting setup point depends on the value of the lead time d which is necessary to prepare a new run (Figure 4.12).

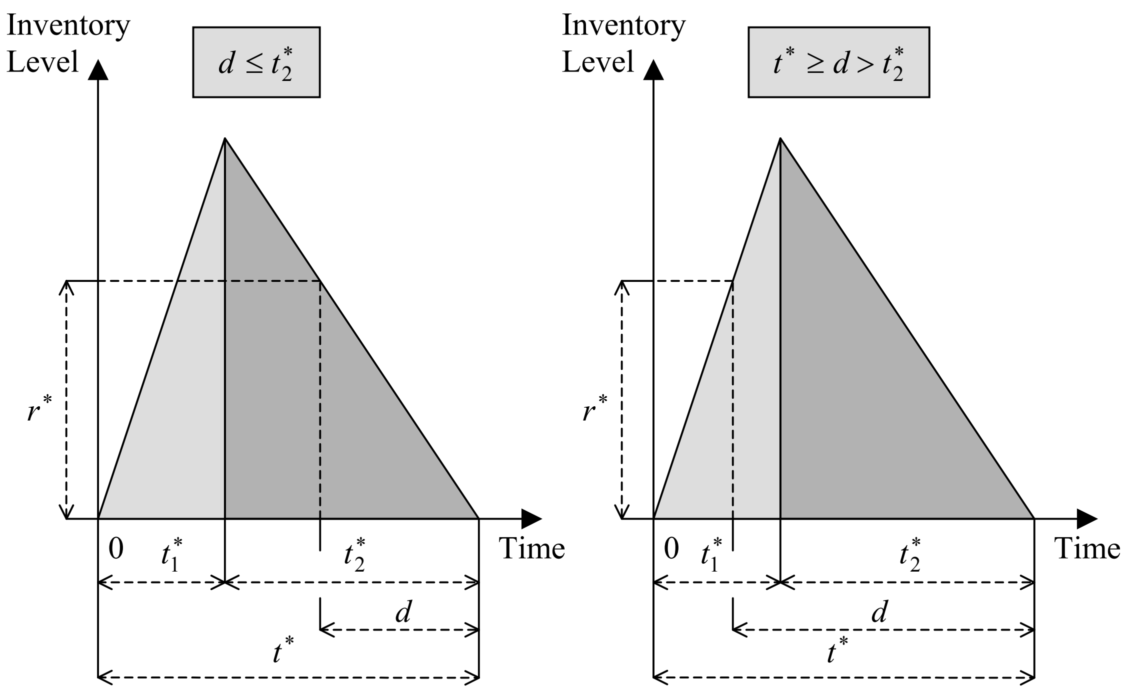


Fig. 4.12 Starting Setup Point ("Reorder Point")

1. If $d \le t_2^*$, the setup is started within the depletion period and the alarm inventory level equals exactly to the quantity demanded during the period d:

$$r^* = hd . \tag{4.45}$$

2. If $t^* \geq d > t_2^*$, the setup is started within the production period and the starting setup point corresponds to the quantity produced (and stored) during the period $t^* - d$:

$$\boxed{r^* = (p-h)\,(t^* - d)\,.} \tag{4.46}$$

Thus, monitoring the inventory level, the management must, in addition, detect whether the production runs or does not.

Since in the example, $d = 1/24 \doteq 0.0417 < t_2^* \doteq 0.0422$, Equation (4.45) must be used for the calculation of the optimum starting setup point:

$$r^* = hd = 5\,000 \text{ cases.}$$

Note: If $d > t^*$, the operator "modulo" must be applied in (4.45) and (4.46), similarly to the EOQ model.

4. Summary

The brewery's management should schedule the cleaning lots of **28 436.16 cases** after **86.5 days**. Each cleaning lot takes **71.1 days**. When a lot is finished, **5 064 cases** are in the store and a depletion period begins. Then the management should monitor the inventory level and when it decreases to **5 000 cases**, a new setup must be started.

Following this optimal schedule, the brewery's total annual cost associated with the cleaning bottles and their holding in the store is approximately **101 280 CZK**.

Note: Since almost all the values are noninteger, the results should be corrected using similar procedure as we presented in Section 4.2.2.

4.3 Probabilistic Inventory Models

In all models described in Section 4.2, deterministic demand and uniform inventory depletion have been assumed. In presented brewery's situation, this assumption seems to be reasonable because of the constant rate of filling bottles. However, in many real situations, both demand and depletion rate cannot be predicted exactly and only a probabilistic distribution of them is available.

In most probabilistic models three following basic questions should be discussed:

1. When to order?
2. How much to order?
3. How much to store in a safety stock?

The first and the second question can be found in deterministic models. Probabilistic models must deal, in addition, with the third question. *Safety stock* is a buffer stock, built up to avoid the inventory shortages that can occur in the situations with uncertain demand or fluctuating depletion rate.

We present two probabilistic inventory models: the model with continuous demand and the single-period decision model.

4.3.1 Probabilistic Model with Continuous Demand

Except for several differences in assumptions, a probabilistic model with continuous demand is rather close to the EOQ model presented in Section 4.2.1. Therefore, sometimes it is called as the EOQ Model with Safety Stock.

Assumptions:
- single item is considered,
- demand for the item is not known with certainty (the probabilistic distribution of demand is given),
- lead time is known and constant over the time,
- continuous, but not uniform depletion of the inventory is supposed,
- unit purchasing cost is independent of the order quantity,
- unit holding cost is independent of the order quantity,
- no additional cost are considered in case of shortage,
- replenishment is executed exactly on the point when the shipment arrives (if shortage occurs, the unsupplied demand is filled immediately on the delivery point).

As Figure 4.13 shows, probabilistic character of demand may cause an overstock (Cycle I) or a stockout (Cycle II):

1. In the first situation a delivery arrives when some inventory remains in the store. In this case the real demand during the lead time is lower than a reorder point (an inventory level when a new reorder is placed).

2. In the second situation a shortage occurs, as the real demand during the lead time exceeds a reorder point.

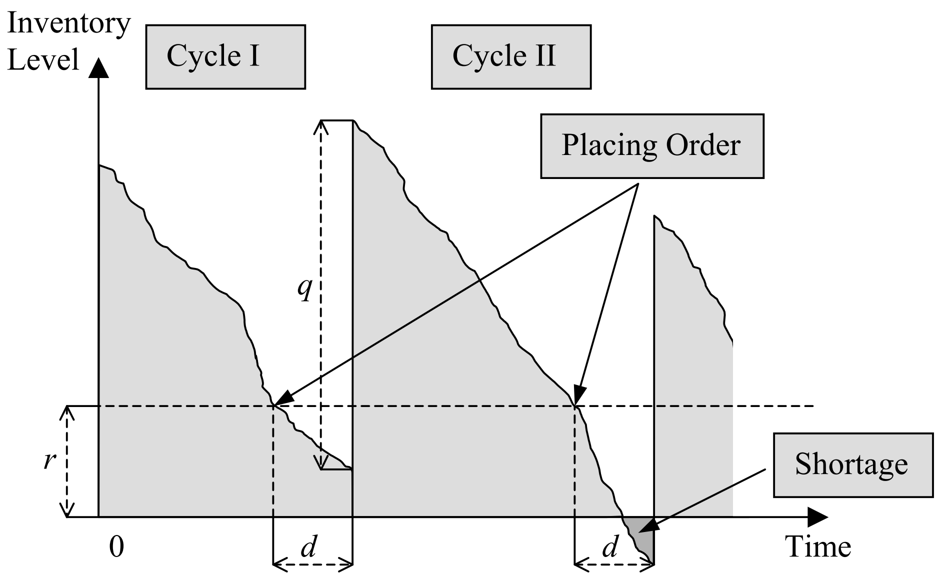


Fig. 4.13 Inventory Level in Model with Probabilistic Demand

To be able to analyze this probabilistic model we must know:

- Type of continuous probability distribution of demand - we can suppose the normal distribution (Figure 4.14).
- Mean of demand μ_Q.
- Standard deviation of demand σ_Q.

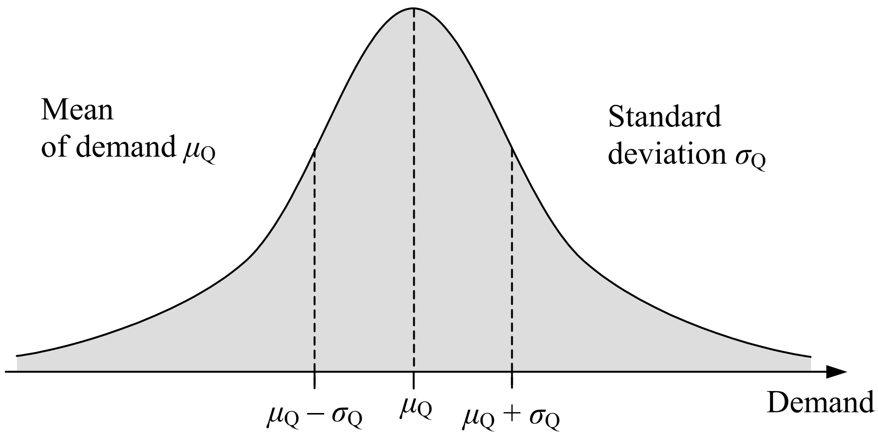


Fig. 4.14 Normal Distribution of Demand

Example 4.6

Suppose now that the brewery's production of beer (in Example 4.2) follows consumers' thirst that depends on probabilistic processes (e.g. weather changes, tourism, etc.). From the company's sale statistics the annual demand is estimated at 120 000 cases and standard deviation at 12 000 cases. The brewery's managers suppose the normal probability distribution of demand. All the other parameters of the model are identical to the parameters in Example 4.2.

The objective is to schedule the inventory process with respect to possible shortages occurrence.

Solution

1. **Calculation of optimum order quantity and optimum reorder point**

For this purpose, we can use the equations from Section 4.2.1, however instead of a deterministic value of demand Q we use the mean of probabilistic demand μ_Q. Since $\mu_Q = 120\ 000$, an estimation of **optimum order quantity** is $q^* = 12\ 000$ cases.

Since the annual demand is a random variable, demand within the lead time (*lead-time demand*) is a random variable as well. The **mean** and the **standard deviation of demand within the lead time *d*** can be expressed as follows:

$$\boxed{\begin{aligned} \mu_d &= d\mu_Q, \\ \sigma_d &= d\sigma_Q. \end{aligned}} \tag{4.47}$$

Hence, the **optimum reorder point** estimation is:

$$\boxed{r^* = \mu_d.} \tag{4.48}$$

Considering in the example $\mu_Q = 120\ 00$ cases, $\sigma_Q = 12\ 000$ cases and $d = 1/24$ of year, the optimum reorder point $r^* = (1/24)\ 120\ 000 = 5\ 000$ cases. The standard deviation of the lead-time demand is $\sigma_d = (1/24)\ 12\ 000 = 500$ cases.

The lead-time demand is supposed to have the normal probability distribution (Figure 4.15), similarly to the total annual demand.

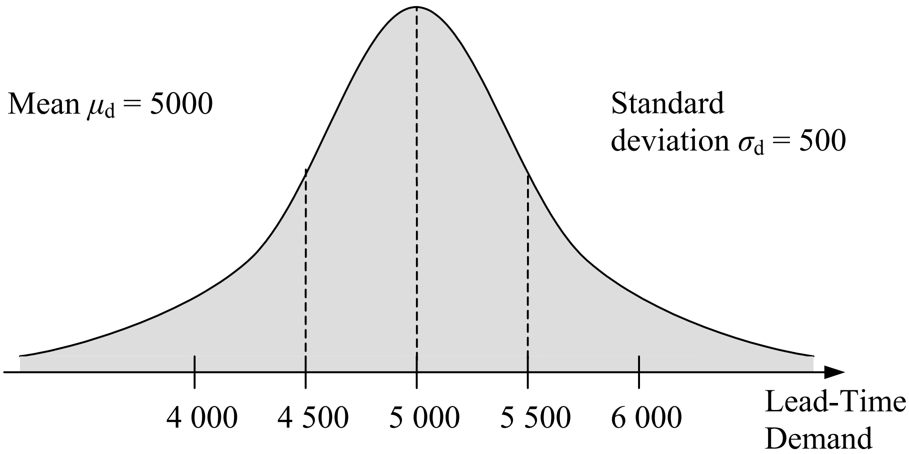


Fig. 4.15 Normal Distribution of Lead-Time Demand

2. Building a safety stock

In Section 4.2.2 the deterministic model with *planned shortages* was introduced. Since in the probabilistic model, demand is uncertain, the *shortages occur randomly* and thus, they cannot be planned. To avoid or reduce a risk of stockout, in most of probabilistic models a *safety stock* is being built up.

For this purpose, we provide three alternative interpretations of the key term in probabilistic models – *service level*:

1. *Service Level* is the probability with which demand will be met within the inventory cycle.
2. *Service Level* is the probability with which shortage will not occur within the inventory cycle.
3. *Service Level* is the percentage of time that all demand is met.

Since the safety stock is built up to protect the company against the shortages, its level depends on the required service level. Suppose 20 planned orders (cycles) and management's decision that only one stockout can be tolerated (5%). Hence, the service level is 95%. The objective is to determine the safety stock that avoids other stockouts. Of course, the higher the safety stock is, the more money a company must pay as the holding cost. Therefore, a compromise decision must be made in such situations.

Building the safety stock actually means a decision to place an order before reaching the optimum reorder point. Thus, the implemented **reorder point** can be expressed as follows:

$$\boxed{r_p = r^* + w,} \tag{4.49}$$

where

r_p … reorder point for the given service level p,
r^* … optimum reorder point (4.48),
w … safety stock level.

As mentioned above, use of the safety stock increases the holding cost, what leads to increasing the **mean of the total cost**:

$$\boxed{\mu_{\mathrm{TC}} = \sqrt{2\mu_{\mathrm{Q}} c_1 c_2} + c_1 w.} \tag{4.50}$$

The objective is now to find such a safety stock level w that satisfies the given service level and minimizes the formula (4.50).

Denoting Q_d as the real lead-time demand, the probability with which this value will be less than or equal to the reorder point r_p, must equal at least the service level p:

$$\boxed{\mathrm{P}\left\{Q_\mathrm{d} \le r_p\right\} \ge p.} \tag{4.51}$$

Note that the probability on the left-hand side of the inequality (4.51) is a probability that the inventory is not totally depleted before a new delivery. This probability, of course, must satisfy the required service level.

According to the initial assumption, the real lead-time demand Q_d has the normal probabilistic distribution with the mean (4.48) and the standard deviation (4.47): $\mathrm{N}\,(r^*, \sigma_\mathrm{d})$. This distribution is transformed into the standard normal distribution N (0, 1) using the following formula:

$$\boxed{z_p = \frac{Q_\mathrm{d} - r^*}{\sigma_\mathrm{d}}.} \tag{4.52}$$

Formula (4.52) can be written in the form:

$$\boxed{Q_\mathrm{d} = z_p \sigma_\mathrm{d} + r^*.} \tag{4.53}$$

After substitution of (4.53) and (4.49) into (4.51) we get:

$$\mathrm{P}\left\{z_p \sigma_\mathrm{d} + r^* \le r^* + w\right\} \ge p,$$

and then, after the simplification:

$$\boxed{\mathrm{P}\left\{z_p \sigma_\mathrm{d} \le w\right\} \ge p.} \tag{4.54}$$

Finally, Equation 4.54 determines the minimum level of the safety stock w that must be built up to satisfy the service level p:

$$\boxed{w \ge z_p \sigma_\mathrm{d}.} \tag{4.55}$$

1. Suppose that the brewery's management decided to keep the service level at 95%. In the table of the standard normal distribution N (0, 1) in Appendix we look up the value z_p corresponding to the given probability $p = 0.95$. The corresponding value found in the table is $z_{0.95} = 1.645$. Thus, for the value of the safety stock we get:

$$w \ge 1.645\ (500) \text{ cases.}$$

Minimizing the mean of the total cost (4.50), the **optimum safety stock level** is:

$$w^* \doteq 823 \text{ cases.}$$

2. If the service level is 99%, the corresponding distribution value is approximately $z_{0.99} = 2.327$ and the **optimum safety stock level** is:

$$w^* = 2.327\ (500) \doteq 1\,164 \text{ cases.}$$

3. Summary

Because of probabilistic demand, the brewery's management must consider the possibility of shortages that may occur within the year.
If managers require **95%** service level, the safety stock of **823 cases** must be built in each cycle, i.e. the reorder point is 5 000 + 823 = **5 823 cases**. Total inventory cost is 240 000 + 20(823) = **256 460 CZK**.

If the service level is being set to **99%**, the safety stock of **1 164 cases** must be built in each cycle and hence the reorder point 5 000 + 1 164 = **6 164 cases**. Total inventory cost is 240 000 + 20(1 164) = **263 280 CZK**.

4.3.2 Single-Period Decision Model

Up to this point we have dealt with the systems operating continuously and having many inventory cycles. The single-period inventory model refers to the situations in which only one order is placed for the considered time period and the inventory is out of stock or there is a surplus of units at the end of that period. There is a penalty associated with overstock or stockout. Rarely, demand is equal exactly to the supply and no penalty is counted.

The single-period models concern seasonal or perishable items that cannot be kept for future periods. As a typical example of such goods we can mention newspapers that do not keep their recency to the next day. Therefore, a problem dealing with the single-period decision model is often known as the *Newsboy Problem*. Other seasonal or perishable goods are, for example, seasonal clothing, Christmas trees, Halloween pumpkins, flowers, bread, some fruits, salads, reserved rooms or seats, etc.

Since in all of the mentioned situations, demand is mostly uncertain, we will develop the *probabilistic* single-period model. To be able to solve such problems, we must know the probabilistic distribution of demand. Knowledge of (either continuous or discrete) the distribution can be stated e.g. from the history (as the statistical values), from the marketing research or analysis, etc. Placing the order results in building the initial inventory at the beginning of the period. Within this period the real demand is less than, greater than or equal to the initial inventory level.

Example 4.7

A bakery department's manager of a new supermarket Happyland should optimize everyday order of rolls. The supermarket buys a roll for 1 CZK and sells it to the final consumers for 2 CZK. If, in the evening some rolls remain in the store, the bakery department changes them into crumbs that will be sold later for 12 CZK per sack. For filling one sack of crumbs 20 rolls are needed.

From the experience of opening comparable supermarkets in comparable areas, the supermarket's analyst recommends to consider the normal probability distribution of the daily demand with a mean of 10 000 rolls and a standard deviation of 500 rolls.

Solution

1. Unit cost calculation

Before specifying the cost we define:

- q – daily quantity of ordered rolls (initial inventory level),
- Q – real daily demand for rolls,
- μ – mean value of daily demand,
- σ – standard deviation of daily demand.

Because of the uncertainty of the daily demand, three situations can occur in the supermarket:

1. The real demand is less than the order quantity ($\boldsymbol{Q < q}$).
   In this case, in the evening $(q - Q)$ rolls remain in the store and they are used for crumbs.

   Unit cost in this situation corresponds to the *marginal loss* that can be expressed as follows:

$$\boxed{ML = \text{purchase cost} - \text{salvage value.}} \tag{4.56}$$

   In the example, the purchase cost is 1 CZK per roll. The salvage value is 0.6 CZK per roll as each roll corresponds to 1/20 of a sack of crumbs (and each sack's value is 12 CZK). Hence the marginal loss per a roll:

$$ML = 0.4\,\text{CZK.}$$

2. The real demand is greater than the order quantity ($\boldsymbol{Q > q}$).
   In this situation, $(Q - q)$ rolls are demanded (by customers) but not sold because of their shortage. Unit cost in this case equals the *marginal profit loss*:

$$\boxed{MPL = \text{selling price} - \text{purchase cost.}} \tag{4.57}$$

   Since, in the supermarket, the selling price is 2 CZK per roll and the purchase cost is 1 CZK per roll, the marginal profit loss per roll due to the unsatisfied demand is:

$$MPL = 1\,\text{CZK.}$$

3. The real demand equals the order quantity ($\boldsymbol{Q = q}$).
   In this ideal, but rather hypothetic situation no loss is generated.

2. Determination of the optimum initial inventory level

If we denote p as the probability with which no stockout occurs, an expected marginal loss per roll is then $p\,(ML)$. Since $(1 - p)$ is the probability of stockout, an expected marginal profit loss per roll is (1 - p) MPL. The following equation corresponds to the optimum expected cost:

$$p(ML) = (1 - p)MPL\,.$$

Hence, the probability with which no stockout occurs can be expressed as follows:

$$\boxed{p = \frac{MPL}{ML + MPL}\,.} \tag{4.58}$$

Similarly to the previous model (see the inequality (4.51)), the initial inventory level q (daily order quantity) must satisfy the following inequality:

$$\boxed{\mathrm{P}\left\{Q \le q\right\} \ge p\,.} \tag{4.59}$$

The probability p in (4.59) represents the *optimum service level* (the probability with which demand will be satisfied within the given period).

Since the demand Q in Happyland has the normal probabilistic distribution with the mean $\mu = 10\,000$ rolls and the standard deviation $\sigma = 500$ rolls, we must transform the general normal distribution N (μ, σ) into the standard normal distribution N (0,1) using known formula:

$$\boxed{z_p = \frac{Q - \mu}{\sigma}\,.} \tag{4.60}$$

Extracting Q from (4.60) and introducing it into (4.59) we get the inequality that must be satisfied to keep the service level p:

$$\boxed{q \ge \mu + z_p\sigma\,.} \tag{4.61}$$

First, in the example, we count the value of the **optimum service level**:

$$p = \frac{1}{0.4 + 1} \doteq 0.7143\,.$$

Then, for this probability we look up the value of the standard normal distribution N (0, 1):

$$z_{0.7143} = 0.566\,.$$

Thus, for the initial inventory level we can write:

$$q \ge 10\,000 + 0.566\,(500) = 10\,283 \text{ rolls.}$$

The **optimum initial inventory level** (the **optimum daily order quantity**) is:

$$q^* = 10\,283 \text{ rolls.}$$

Suppose now that the discrete probabilistic distribution of the daily demand Q is available instead of the description in the form of the continuous distribution. In Table 4.5, the probability of each demand level is estimated. In addition, cumulative probabilities (with which demand is less then or equal to the given level) are calculated.

Demand Level	Probability	Cumulative Probability
8 000	0.01	0.01
8 200	0.02	0.03
8 400	0.04	0.07
8 600	0.06	0.13
8 800	0.07	0.20
9 000	0.09	0.29
9 200	0.10	0.39
9 400	0.11	0.50
9 600	0.12	0.62
9 800	0.10	0.72
10 000	0.09	0.81
10 200	0.07	0.88
10 400	0.05	0.93
10 600	0.04	0.97
10 800	0.02	0.99
11 000	0.01	1.00

Tab. 4.5 Discrete Probabilistic Distribution for Demand

Since the calculated value of the service level $p \doteq 0.7143$ falls into the interval <0.62; 0.72> in the column of the cumulative probabilities, the **optimum daily order quantity** can be found in the interval <9 600; 9 800> rolls. If the manager may order rolls in boxes of 200 rolls (i.e. 150 rolls, for example, cannot be ordered), the optimum order quantity will be set at the level:

$$q^* = 9\,800 \text{ rolls.}$$

3. Summary

When the bakery department's manager has the above-mentioned information about the continuous probability distribution of demand, and wants to keep the service level 0.7143, the optimum initial inventory level is **10 283 rolls**.

Considering the table of the discrete probabilistic distribution of demand, the manager should order **9 800 rolls** each day.

4.4 Glossary

Cycle Time – dodávkový cyklus
The length of time between the placing of two consecutive orders.

Demand Rate – intenzita poptávky; intenzita spotřeby
Inventory demanded within a specific time period.

Depletion – čerpání zásoby; vyčerpání zásoby
The process of reduction of the inventory, or reduction of the inventory to a zero point.

Deterministic Inventory Model – deterministický model zásob
Inventory model in which demand within a time period is known with certainty.

Economic Order Quantity (EOQ) – optimální velikost objednávky
An order quantity that minimizes the annual holding cost plus annual ordering cost.

Holding Cost – skladovací náklady
Variable cost associated with storing inventory.

Inventory Level – stav zásoby
The current (available) amount of inventory.

Lead Time – pořizovací lhůta dodávky
The time between placing an order and receiving the shipment (delivery).

Marginal Loss – mezní ztráta
Unit cost (purchase cost minus salvage value) that occurs when demand is less than the order quantity.

Marginal Profit Loss – mezní ušlý zisk
Unit loss (selling price minus purchase cost) that occurs when demand is greater than the order quantity.

Ordering Cost – pořizovací náklady
Fixed cost of placing one order.

Probabilistic Inventory Model – stochastický model zásob
Inventory model in which demand fluctuates throughout time period and it is described as a random variable.

Production Lot Size – velikost výrobní dávky
Amount of items being produced in one cycle.

Production Rate – intenzita výroby
Number of items being possibly produced within specific time period.

Quantity Discount – množstevní rabat
A discount on the unit purchasing cost offered by supplier to a buyer willing to buy in large lots.

Reorder Point – bod znovuobjednávky
The inventory level at which a new order is placed.

Safety Stock – pojistná zásoba
Inventory maintained specifically to reduce shortages.

Service Level – úroveň obsluhy
The probability with which demand is met within the inventory cycle. The percentage of time that all demand is met on request.

Setup Cost – náklady na přípravu výrobní dávky
Fixed cost associated with preparation of a production run (lot).

Shortage (Stockout) – nedostatek zásoby
Inability to provide the units from stock. Available inventory is insufficient to meet demand.

Shortage (Stockout) Cost – náklady z nedostatku zásoby
Variable cost associated with shortage of inventory.

Surplus – nadbytek zásoby
Available inventory exceeds demand.

Unit Purchasing Cost – nákupní cena
Variable cost associated with purchasing a single unit of the inventory (unit price).

5. Waiting Line Models

In this chapter we deal with the real-life situations in which *customers* come to *service system* with their requirements in order to be fully satisfied. In general, both the customers and the *servers* in service systems may be people, machines, jobs etc. In the following table you can find several examples of such situations.

Service System	Customer	Server
Doctor's consultancy room	Patient	Doctor
Service shop	Machine	Mechanic
Bank	Client	Clerk
Crossing	Car	Traffic lights
Telephone exchange	Call	Switchboard
Airport	Airplane	Runway
Airport	Passenger	Passport control
First-aid station	Car accident	Ambulance
Fire station	Fire	Emergency unit
Service station	Car	Petrol pump
Restaurant	Consumer	Seat
World Wide Web	PC station	Web server
PC station	Job	Control unit

Tab. 5.1 Examples of Waiting Line Systems

Because of server's limited capacity, customers may wait until the service of the customers who came before is finished. Since, in presented systems, *queues* are typically formed, the waiting line models are sometimes called *queuing models* and the used theory is called *queuing theory*.

5.1 Introduction to Queuing Theory

As mentioned above, in each waiting line system two basic elements are distinguished: a *customer* and a *server* (*service facility*). Figure 5.1 shows the general structure of a queuing system.

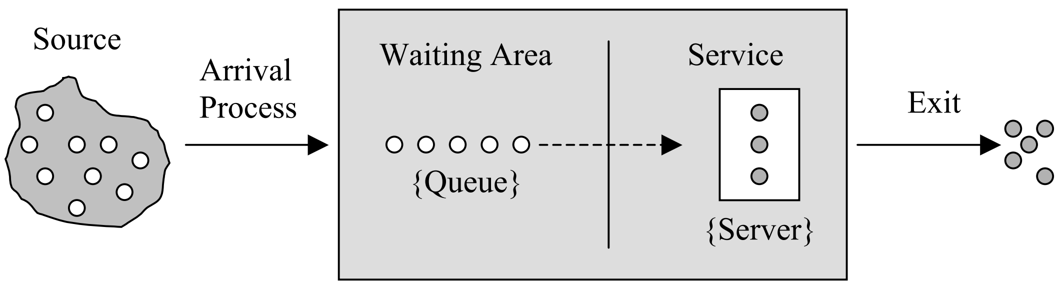


Fig. 5.1 Waiting Line System

The **source** represents the group of potential customers (e.g. the whole population). When some of them require a service, they become real **customers** who enter the system through the **arrival process**. If the **server** providing the required **service** is available, the customer is served immediately. Otherwise, the customer has to wait in a **queue** until the server finishes the service of the previous customers. After finishing the service, the customer **exits** the system.

5.1.1 The Arrival Process

The **source** of customers might be either *finite* of *infinite* (unlimited). Actually, no source is really infinite, however we may consider the source as infinite (for the system) if it is so large that the probability of a second arrival is not significantly changed by the first arrival. As the example of such customers we can mention e.g. the visitors of Prague Castle. The finite source may be, for example, the enginery in a factory (limited number of machines). Customers may enter the system either in *batches* (goods' delivery, passengers at the airport gate, castle visitors, etc.) or *individually* (patients, cars in a car-wash line, train arrivals, etc.).

Arrivals of customers may be *scheduled* (trams coming to a tram stop) or *unscheduled* (patients requiring the emergency aid). A time period between arrivals of two sequential customers is called the *interarrival time*. If customers‘ arrivals are monitored through any time unit (hour, day, etc.), the *average interarrival time* can be counted. The reciprocal value of the average interarrival time is the *average arrival rate* (the average number of arrivals per time unit). In case of unscheduled arrivals, the average arrival rate is a random variable, mostly described by a probability distribution.

For appropriate description of the arrival process the *Poisson discrete probability distribution* may be used. It describes the average number of customers coming per specific time unit (e.g. per hour). The mean of this probability distribution is denoted λ. The interarrival times then follow the continuous *exponential distribution* with the mean $1/\lambda$. Both distributions are possible to be used in case of independent arrivals of the customers, i.e. if the arrival of a customer is independent of the arrival of previous customers and simultaneously does not significantly affect the future arrivals.

5.1.2 The Service Process

Similarly to the interarrival time between coming customers, we are interested in the **time** the customer spends at the service facility, called *service time*, which may be either *deterministic* or *probabilistic*. In probabilistic (fluctuating) case, the length of the service action mostly follows the *exponential distribution*. The *average service time* is usually designated by mean time $1/\mu$, where μ determines the *average service rate* (it measures the average server's capacity – number of customers being possibly served per time unit). From the above consideration, it ensues that the service rates follow the *Poisson distribution*.

The service part of the waiting lines system (Figure 5.1) is determined by the **type, number** and **arrangement** of the service facilities. We offer basic possibilities of the service configurations:

1. *Single facility*. Only one server is available to serve the customers (e.g. dentist's chair, shop assistant behind the shop board).

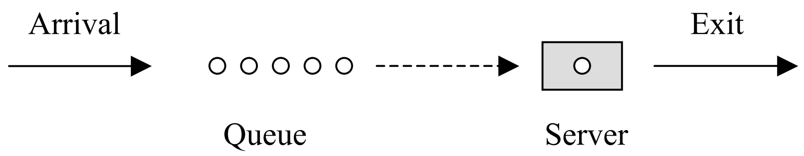


Fig. 5.2 Single Service Facility

2. *Multiple*, *parallel*, *identical facilities*. In the service system a customer can find several servers providing identical service within identical average service time. Either *single queue* or *multiple queues* may be considered in such systems. Whereas in case of single queue (Figure 5.3) the customer mostly has to respect the rule: go to the first available server (e.g. bank counters, shop assistants behind the shop board, ambulance), in case of multiple queues (Figure 5.4) customers are allowed to choose the server (e.g. with the shortest queue) by themselves (petrol pumps in a gasoline station, checkout counters in a supermarket, carriageway at traffic lights).

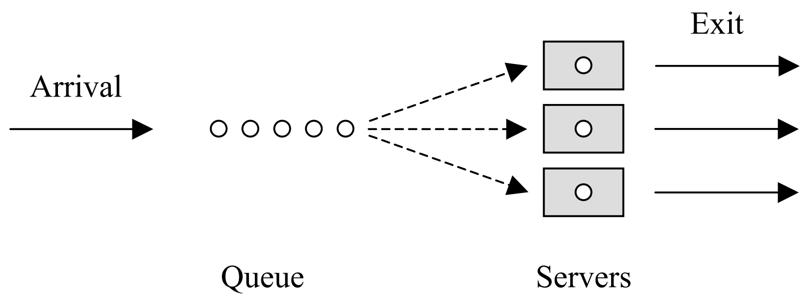


Fig. 5.3 Parallel, Identical Service Facilities (Single Queue)

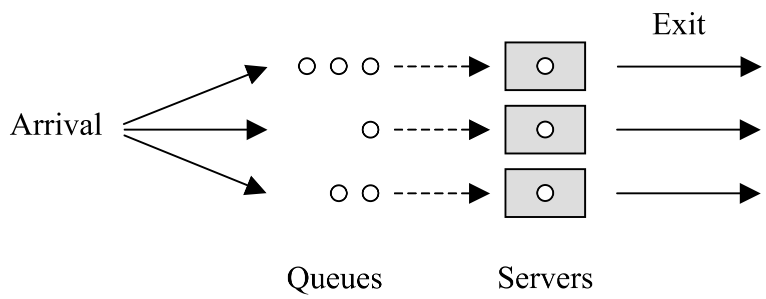


Fig. 5.4 Parallel, Identical Service Facilities (Multiple Queues)

3. *Multiple*, *parallel*, but *not identical facilities*. Although in many situations servers provide the same service (in terms of satisfying identical needs), they may differ, for example, in the value of the average service time (e.g. express and regular mechanics) or in the method of payment (cash, credit card, cheque), in the way of service (declare or nothing

to declare in the airport), etc. In such situations the customers must (or may) choose the server (queue) and therefore multiple queues are typically formed.

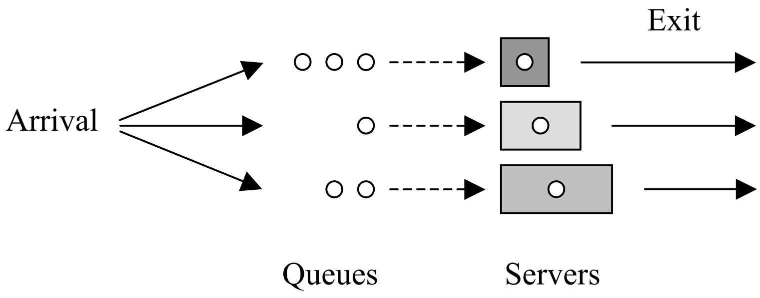


Fig. 5.5 Parallel, Nonidentical Service Facilities (Multiple Queues)

4. *Serial facilities*. Customer must go through all the facilities arranged in a series. The servers, of course, are seldom identical. Good example is the tour of the patient with the broken leg to be treated in the policlinic. First, he must wait for a doctor who sends him to the X-ray photography. Then the patient waits both for taking photography and for making it. When he turns back to the doctor he may wait until the previous patient is served, and after the doctor decides that the leg must be fixed in the plaster, patient is taken to the special department for making the plaster bandage. In case the patient cannot walk, he must wait for the ambulance car to take him home. On the tour the patient goes through many service facilities and must queue quite often.

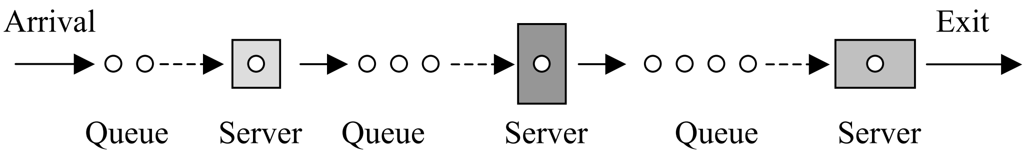


Fig. 5.6 Serial Service Facilities

5. *Combination of facilities*. The basic configurations described above may be combined into a complex structure of service facilities. If, in the previous example of the patient's tour, the whole hospital were considered instead of the single orthopedics, the complexity of such waiting line system would be even more evident.

5.1.3 The Waiting Line

Waiting area (Figure 5.1) is the part of the system where customers must spend some time when the service facility is busy. After finishing the service of the customer coming before, next customer leaves the waiting area and proceeds to the server. In order to make clear who is the next, we must understand the **discipline of the queue** being formed in the waiting area:

1. *FCFS (First-come, first-served)*. Customers are served in the same order they have arrived, i.e. the first-coming customer is served the first from all who are waiting in the queue. An alternative name of this queue discipline is *FIFO (First-in, first-out)*. This rule is typical for most of the waiting line systems (shops, banks, hospitals, tram stops, printers, traffic lights,

service stations, some assembly lines, etc.) and it is assumed in the models described in the following sections.

2. *LCFS (Last-come, first-served)*, or *LIFO (Last-in, first-out)*. Last arrivals are served first. This queue discipline can be found, for example, in warehouses or in production systems, where it enables to reduce handling and transportation.

3. *PRI (a priority system)*. In many situations customers may use the advantage of the priority. It is a predefined rule that determines the order in which customers proceed to the server. For example, handicapped or old people, children and women may be selected to precede the young men at fire escape. People living in dangerous areas may have higher priority to the others in case of vaccination against icterus. In many situations, V.I.P. (Very Important People) have the highest priority.

   In e*mergency (preemptive priority) systems*, the importance of an arriving customer may be so high that the service process of a less important customer can be even interrupted. For example, if a serious problem occurs, the doctor-expert may be withdrawn from the less important job. Inspector will probably interrupt questioning the booster after the bank robbery has been suddenly announced.

   If people with exactly the same level of priority are proceeding to the server (e.g. the same-aged children), a further rule must be used for the selection of the first of them (e.g. FCFS).

4. *SIRO (Selection in random order)*. There is no predefined order of service, i.e. customers are selected randomly. For example, at the press conference the chairperson (mostly) selects the journalists' questions without any specific rule, passengers on sinking boat must draw lots to decide the order to embark the lifeboats, etc.

5.1.4 Analysis of Waiting Line Models

The structure and attributes of the waiting line systems have been discussed. However, managers are interested in optimizing the system's behavior. In order to be able to expertise the quality of the system (e.g. to find weaknesses in the waiting line system), the manager requires the essential cost and other characteristics describing performance of the model.

Cost consideration

Similarly to other managerial situations, the objective of decision making in waiting line models is to minimize the total cost. Although the manager does not operate on the customers' side, he should be interested in their satisfaction with the service process. The increasing *service level* may lead to an increase in profit, or to a decrease in *waiting cost*. From this point of view the successful manager determines the number of servers to assure full satisfaction of the customers. However, this decision sharply increases the *service cost*. For example, if there are too long queues in a supermarket in front of the checkers, customers would prefer an alternative supermarket where they would be served without waiting. In this case the managers should evaluate the profit loss and consider a decision of hiring additional checkers to reduce waiting time. Of course, additional wages (fixed cost) are connected with this decision.

The *service cost* (*facility cost*) generally includes:

- *Cost of construction* (capital investment). When, e.g. a new production line is being constructed, all the fixed cost connected with this project must be considered.
- *Cost of operation*. This variable cost includes material, labor and energy cost required for operations (service providing). In case of the mentioned production line the company must consider operator's wages, and power or fuel consumption.
- *Cost of maintenance and repair*. Maintenance of the production line and all its repairs must be calculated.
- *Other costs*. For example, insurance and rental of space for the production line must be paid.

Similarly to the inventory models, the compromise must be found in waiting line models as well.

Time characteristics

When a customer is making a decision on entrance the waiting line system he may be interested in the total time he would spend in the system. Because of probabilistic character of the models only estimations or average values can be calculated. Two basic time measures of the system's performance are:

- *average waiting time in the queue*,
- *average waiting time in the system* (time spent waiting for the service and being served).

Of course, not only customers are interested in these two characteristics, but also managers who are responsible for the optimal performance of the system.

Number of customers

There are two other important estimations, evaluating the behavior of the analyzed system:

- *average number of customers in the queue*,
- *average number of customers in the system* (i.e. customers in the queue and in the service process).

Probability characteristics

- *probability of an empty service facility* is the probability that there are no customers in the system (the facility is *idle*),
- *probability of the service facility being busy* equals the *probability that an arriving customer has to wait for service* (e.g. in case of system with the single server it is the probability of not finding an empty system); this measure refers to the utilization of the service facility,
- *probability of finding exactly N customers in the system* (waiting in the queue or being served),
- *probability that the number of customers in the system (N) will be larger than a specified number (n)*,
- *probability of being in the system longer than time t*; this characteristic might be, from the customer's point of view, rather important in the situation where several alternative services are being offered.

5.2 Basic Waiting Line Models

As the computation of all the measures mentioned in the previous section depends on the structure of the analyzed waiting line system, we will first outline the models *classification* in this section. Then we will present two basic waiting line models: *standard single-server model* and *standard multi-server model*.

5.2.1 Classification of Waiting Line Models

The Kendall's notation uses the standard sequence of symbols for the waiting line models classification:

A/B/C/D/E/F.

Each of the six positions in the pattern corresponds to one of six model's attributes. All together enable perfect description of basic waiting line systems. The meaning of the attributes is following:

A describes the arrival process, i.e. the type of the probability distribution corresponding to the interarrival time. For example, symbol **M** in this position is used for the exponential distribution, **D** for constant time period, **U** for the uniform probability distribution, **N** for the normal distribution etc.

B refers to the probability distribution of the service time. Exactly the same symbols are used as for the distributions of the interarrival time.

C determines the number of parallel servers.

D is used for the queue discipline (**FCFS**, **LCFS**, **PRI**, **SIRO**).

E is the maximum length of queue. If there is no limit for the queue, symbol ∞ is used, otherwise given number *n* is written in this position.

F specifies the size of the customers' source. For the infinite source we use ∞, whereas in case of the finite source we write the given number *n*.

Using the Kendall's notation we offer several examples of the most common waiting line systems:

M/M/1/FCFS/∞/∞	standard single-server model,
M/M/K/FCFS/∞/∞	standard multi-server model,
M/M/1/PRI/∞/∞	priority service, single-server model,
M/M/1/FCFS/*n*/∞	limited queue, single-server model,
M/M/K/FCFS/∞/*n*	finite source, multi-server model.

Note: If only A/B/C notation is used for the model, it means that FCFS queue discipline, no limit for the queue and infinite source are assumed.

5.2.2 Standard Single-Server Exponential Model M/M/1

The simplest model (shown in Figure 5.2) among waiting line models is predicated on the following assumptions:

Assumptions of M/M/1 model:
- waiting line has a single server,
- interarrival times are described by exponential probability distribution with the mean = $1/\lambda$,
- service times follow exponential probability distribution with the mean = $1/\mu$,
- infinite source,
- unlimited length of queue,
- the queue discipline is FCFS.

As described in Section 5.1, all waiting line models can be characterized by several measures of performance. We do not present the derivation of all the formulas because of its complicacy. We use the following example to illustrate the calculation of the measures.

Example 5.1

In the small grocery store there is only one shop board with only one shop assistant. In the period from 8 a.m. to 6 p.m. 18 customers per hour (on the average) come to the grocery. The assistant is able to serve (on the average) 25 customers per hour. The assistant's salary is 15 000 CZK per month. The grocery's owner tested another assistant who was able to serve 29 customers per hour. Because of faster service in this case 20 customers per hour (on the average) came to the grocery. However, this assistant requires the salary of 22 000 CZK per month. The average profit is 50 CZK per sale. The grocery's owner considers hiring the faster assistant instead of the slower one.

Solution

Since it will be quite interesting to compare both alternatives to each other step by step, we will calculate all the characteristics simultaneously. All noninteger values are rounded to appropriate number of decimal figures.

1. **Definition of input parameters**

The average arrival rate λ

The number of coming customers follows the Poisson probability distribution (the interarrival times are described by the exponential distribution). Whereas in the first alternative the average arrival rate is **18 customers per hour**, in the second alternative it is **20 customers per hour**.

	Slow Assistant	Fast Assistant
λ	18	20

The average service rate μ

The service times follow the exponential probability distribution. The average service rate is the reciprocal value of the average service time. In the example, the average service rates are **25** and **29 customers per hour**.

	Slow Assistant	Fast Assistant
μ	25	29

The salary
The monthly salaries of the assistants are **15 000** and **22 000 CZK**.

	Slow Assistant	Fast Assistant
S	15 000	22 000

The sale profit
The average sale profit is the same for both the assistants: ***SP* = 50 CZK per sale**.

2. Calculating measures of performance

The utilization of the system
This characteristic is calculated as the ratio of the average arrival rate and the average service rate:

$$\rho = \frac{\lambda}{\mu}. \tag{5.1}$$

If $\rho \geq 1$, the length of the queue would increase without any bound. Therefore only the assumption of $\rho < 1$ is acceptable to management.

	Slow Assistant	Fast Assistant
ρ	0.72	0.69

The utilization of the system is **the probability that the server is busy**, or **the probability that there is at least one customer in the system**.

The probability of an empty facility (server is idle)
Considering this value as the probability that there are no customers in the system, we can use Equation 5.1:

$$P(0) = 1 - \rho\,. \tag{5.2}$$

In the example, the values are computed as follows:

	Slow Assistant	Fast Assistant
$P(0)$	0.28	0.31

The average waiting time in the system
This is an estimation of time a customer spends in the system (waiting for the service and being served):

$$W = \frac{1}{\mu - \lambda}. \tag{5.3}$$

	Slow Assistant	Fast Assistant
W (hours)	0.143	0.111
W (min)	8.6	6.7

This characteristic may affect the customers in their decision to enter the grocery. In case of the faster assistant the customers save (on the average) almost 1/4 of their time (spent in the grocery) in comparison with the slower assistant.

The average waiting time in the queue

This is the average time a customer waits in the queue before the service starts:

$$W_q = W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)}. \tag{5.4}$$

It is quite easy to calculate this measurement, if we consider that the average waiting time in the system W is the average waiting time in the queue W_q, plus the average service time $1/\mu$.

	Slow Assistant	Fast Assistant
W_q (hours)	0.103	0.077
W_q (min)	6.2	4.6

Whereas in the first situation the customers wait in the queue more than 6 minutes (on the average), in the second situation they wait less than 5 minutes.

The average number of customers in the system

Generally, all the customers in the system are those being served together with those who are waiting in the queue. The number of customers in the system can be calculated as the average arrival rate (the number of arrived customers per time unit), multiplied by the average waiting time (average time that the arrived customers spend in the system):

$$L = \lambda W = \frac{\lambda}{\mu - \lambda}. \tag{5.5}$$

	Slow Assistant	Fast Assistant
L	2.57	2.22

The average number of customers in the queue

Similarly to the previous calculation, the number of customers in the queue is the average arrival rate (the number of arrived customers per time unit), multiplied by the average waiting time in the queue:

$$L_q = \lambda W_q = \frac{\lambda^2}{\mu(\mu - \lambda)}. \tag{5.6}$$

	Slow Assistant	Fast Assistant
L_q	1.85	1.53

The probability of finding exactly *N* customers in the system

When N customers are in the system, one customer is being served and (N-1) customers are waiting in the queue.

$$P(N) = P(0)\rho^N = (1 - \rho)\rho^N. \tag{5.7}$$

In the following table we present the probabilities of finding 0, 1, 2, 3, 4, 5 customers in the grocery:

	Slow Assistant	Fast Assistant
$P(0)$	0.280	0.310
$P(1)$	0.202	0.214
$P(2)$	0.145	0.148
$P(3)$	0.105	0.102
$P(4)$	0.075	0.070
$P(5)$	0.054	0.048

The probabilities $P(0)$, $P(1)$ and $P(2)$ are lower for the slow assistant, since in this case fewer customers come to the grocery. However, further probabilities $P(3)$, $P(4)$ and $P(5)$ are lower for the fast assistant because of his perfect service ability (Figure 5.7).

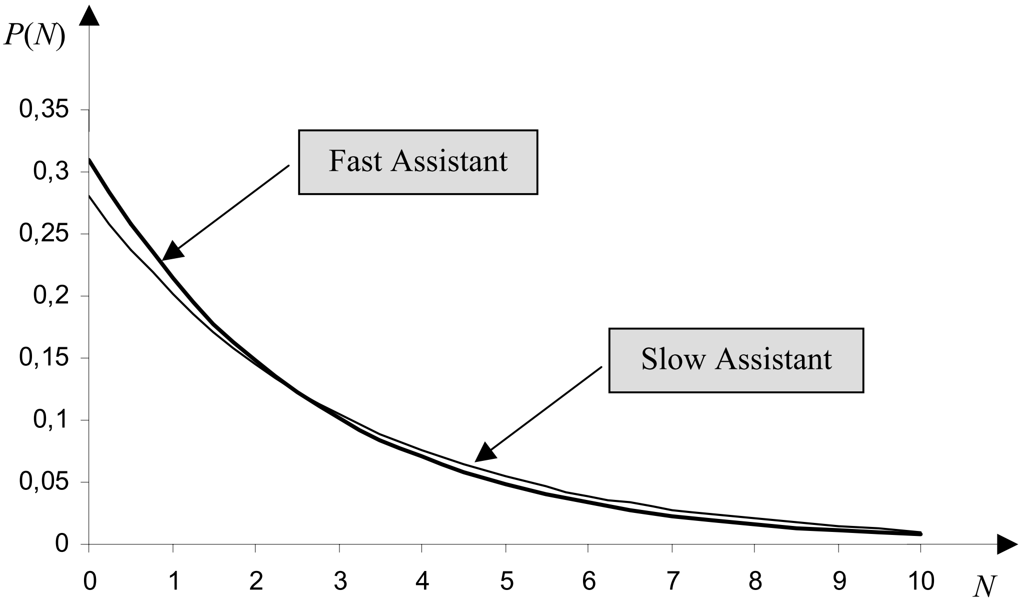


Fig. 5.7 The probability of finding N customers in the system

The probability that the number of customers in the system (N) will be larger than a specified number of customers (n)

$$P\{N > n\} = \rho^{n+1}. \tag{5.8}$$

Similarly to the previous characteristic we draw the following table:

	Slow Assistant	Fast Assistant
$P\{N > 0\}$	0.720	0.690
$P\{N > 1\}$	0.518	0.476
$P\{N > 2\}$	0.373	0.328
$P\{N > 3\}$	0.269	0.226
$P\{N > 4\}$	0.193	0.156
$P\{N > 5\}$	0.139	0.108

The probability $P\{N > 0\}$ is the probability that the system is not empty, i.e. that the server is busy. With the probability $P\{N > 0\}$ the coming customer will be waiting in the queue.

The probability of being in the system longer than specified time (*t*)

Time spent in the system (T) consists of the waiting time in the queue and the service time. The probability that this time is greater than specified time (t) can be expressed as follows:

$$P\{T > t\} = e^{(\lambda-\mu)\,t}. \tag{5.9}$$

In the following table the probabilities (5.9) are computed for several specified times.

	Slow Assistant	Fast Assistant
$P\{T > 1\text{ min}\}$	0.890	0.861
$P\{T > 2\text{ min}\}$	0.792	0.741
$P\{T > 3\text{ min}\}$	0.705	0.638
$P\{T > 4\text{ min}\}$	0.627	0.549
$P\{T > 5\text{ min}\}$	0.558	0.472

Note: Although all times in the table are expressed in minutes, the calculation of (5.9) must respect the chosen time unit (hour, in our case). Thus, parameter t in the formula must be 1/60, 2/60, etc.

3. Cost and profit analysis

If the grocery's owner considers the replacement of the slower assistant, he should be interested in the number of sales, which determines the daily profit. Considering 22 working days within a month (each day has 10 working hours), simple formula can be used for the calculation of total monthly profit:

$$TP = (22)(10)\lambda SP,$$

where

TP … the total monthly profit,
λ … the average number of customers coming to the grocery per hour,
SP … the average profit per sale.

In the following table monthly assistants' salary S can be found together with the calculated total monthly profit TP.

	Slow Assistant	Fast Assistant
S	15 000 CZK	22 000 CZK
TP	198 000 CZK	220 000 CZK

4. Summary

The grocery's owner should hire the fast assistant, as he increases the monthly profit of **22 000 CZK** and required salary will be higher only by **7 000 CZK**. Although the cost and profit analysis is quite simple, several interesting measures of performance have been calculated. We present some of them:

Characteristic	Slow Assistant	Fast Assistant
Utilization of the grocery	0.720	0.690
Probability that the assistant is idle	0.280	0.310
Average waiting time	8.6 min	6.7 min
Average waiting time in the queue	6.2 min	4.6 min
Average number of customers in the grocery	2.57	2.22
Average number of customers in the queue	1.85	1.53
Probability of being in the grocery more than 1 min	0.890	0.861

As it is apparent from the table, the fast assistant improves in total the performance of the grocery and its attractiveness for the customers.

5.2.3 Standard Multi-Server Exponential Model M/M/K

In the previous model the single server has been assumed. However, many waiting line systems (e.g. petrol station, bank, supermarket) have more than one server (e.g. petrol pumps, counters). For this purpose, we add in the waiting line model further parallel servers providing the identical service. In such a model, the following assumptions are:

Assumptions of M/M/K model:

- system has K parallel, identical servers,
- one waiting line (queue) exists,
- interarrival times are described by exponential probability distribution with the mean = $1/\lambda$,
- service times of each server follow exponential probability distribution with the mean = $1/\mu$,
- infinite source,
- unlimited length of queue,
- the queue discipline is FCFS.

The assumption of the only one queue for all servers corresponds to the model in Figure 5.3. Customers who wait in the queue move to the first available service facility. This situation is typical for bank counters, shop assistants, seats in a restaurant, etc.

As μ is the average service rate for each server, $K\mu$ is the average service rate for the system. Similarly to the single-server model, the constraint $\lambda/(K\mu) < 1$ prevents the system from creating the infinite queue.

Example 5.2

We modify Example 5.1. Suppose now that the grocery's owner considers employing two assistants instead of one. Although the shop board is divided into two separate parts, the only one waiting area is in the shop. The customers proceed from the queue to the first available assistant. The second assistant is supposed to work at the same ability level with the first shop assistant, i.e. he is able to serve 25 customers per hour. Therefore his salary is 15 000 CZK per month as well. Because of the faster (doubled) service more people come to the grocery. The estimation of the average arrival rate is 30 customers per hour.

Solution

The derivation of all equations is, of course, more complex for the multiple-server model than for the single-server model, and the equations themselves are more complex at all. We will calculate all the values for the two-assistant grocery and in the summary we will compare them with the results from the single-assistant model.

1. Definition of input parameters

The average arrival rate *λ*
The estimation of this value is ***λ*** **= 30 customers per hour**.

The average service rate *μ*
This value is the average number of customers that each assistant is able to serve per one hour. Ensuing from the definition of the problem, the average service rate is ***μ*** **= 25 customers per hour**.

The number of the assistants *K*
As in the grocery two assistants should be working, ***K*** **= 2**.

The salary
Since the monthly salary of one shop assistant is **15 000 CZK**, the total salary of two assistants is **30 000 CZK**.

	One Assistant	Two Assistants
S	15 000	30 000

The sale profit
The average sale profit is the same both for the single-assistant and for the two-assistant grocery: ***SP*** **= 50 CZK per sale**.

2. Calculating measures of performance

The average service rate of the system *Kμ*
This characteristic is the average number of customers that the system (all the servers) is able to serve per time unit.

In the example, two assistants (together) can serve $K\mu = 2(25) = 50$ customers per hour.

The utilization of the single-server system
This measure is the ratio of the average arrival rate and the average service rate of the server:

$$\boxed{\rho = \frac{\lambda}{\mu}.} \qquad (5.10)$$

The utilization is $\rho = 1.2$.

The utilization of the system
This characteristic can be expressed as the ratio of the average arrival rate and the average service rate of the system:

$$\boxed{\bar{\rho} = \frac{\rho}{K} = \frac{\lambda}{K\mu}.} \qquad (5.11)$$

The utilization of the grocery is $\overline{\rho} = 0.6$. Since this value is less than 1, formation of the infinite queue is avoided.

The utilization of the system is **the probability that all the servers are busy**, or **the probability that there are at least *K* customers in the system**.

The probability of finding no customers in the system (all servers are idle)

$$P(0) = \frac{1}{\dfrac{\rho^K}{K!(1-\overline{\rho})} + \sum_{i=0}^{K-1} \dfrac{\rho^i}{i!}} . \tag{5.12}$$

The probability of finding empty grocery is $P(0) = 0.250$.

The probability of finding exactly *N* customers in the system
Three possible situations may occur in the system:

1. $N < K$. All customers are being served and $(K - N)$ servers are idle.
2. $N = K$. All customers are being served and all servers are busy.
3. $N > K$. All servers are busy and $(N - K)$ customers are waiting in the queue.

Hence, two different formulas are derived for this characteristic:

$$P(N) = \begin{cases} P(0)\dfrac{\rho^N}{N!} & \text{when } N \le K \\ P(0)\dfrac{\overline{\rho}^N K^K}{K!} & \text{when } N > K \end{cases} . \tag{5.13}$$

The following table presents the probabilities of finding 0, 1, 2, 3, 4, 5 customers in the grocery:

	Slow Assistant
$P(0)$	0.250
$P(1)$	0.300
$P(2)$	0.180
$P(3)$	0.108
$P(4)$	0.065
$P(5)$	0.039

Note: The probability of waiting in the queue can be expressed, in the example, as follows:

$$P\{N > 1\} = 1 - (P(0) + P(1)) .$$

Thus $P\{N > 1\} = 1 - (0.250 + 0.300) = 0.450$.

The average number of customers in the queue

$$L_q = P(0)\frac{\rho^K \overline{\rho}}{K!(1-\overline{\rho})^2} . \tag{5.14}$$

In the example, the average number of customers in the queue is $L_q = 0.675$.

The average number of customers in the system

$$L = L_q + \rho \,. \tag{5.15}$$

The average number of customers in the grocery is $L = 1.875$.

The average waiting time in the queue

This is the average time a customer waits in the queue before the service starts:

$$W_q = \frac{L_q}{\lambda} \,. \tag{5.16}$$

The customers wait in the queue (on the average) $W_q = 0.0225$ hours i.e. 1.35 minutes.

The average waiting time

This is the estimation of time a customer spends in the system (waiting for the service and being served):

$$W = \frac{L}{\lambda} = W_q + \frac{1}{\mu} \,. \tag{5.17}$$

The customers spend in the grocery (on the average) $W = 0.0625$ hours i.e. 3.75 minutes.

3. Cost and profit analysis

Similarly to Example 5.1 we present the calculation of cost and profit both for the single-assistant model and for the two-assistant model. The total monthly profit can be expressed as follows:

$$TP = (22)(10)\lambda SP \,,$$

where

TP … the total monthly profit,
λ … the average number of customers coming to the grocery per hour,
SP … the average profit per sale.

In the following table the monthly salary S can be found together with the calculated total monthly profit TP:

	One Assistant	Two Assistants
S	15 000 CZK	30 000 CZK
TP	198 000 CZK	330 000 CZK

4. Summary

Regarding the cost and profit analysis, there is no doubt about employing two assistants instead of one. The comparison of both alternatives in terms of their basic characteristics seems to be more interesting:

Characteristic	One Assistant	Two Assistants
Utilization of the grocery	0.720	0.600
Probability that the grocery is empty	0.280	0.250
Average waiting time	8.6 min	3.8 min
Average waiting time in the queue	6.2 min	1.4 min

Average number of customers in the grocery	2.57	1.88
Average number of customers in the queue	1.85	0.68
Probability of waiting in the queue	0.720	0.450

5.3 Computer Simulation in Waiting Line Models

In many real situations it is entirely impossible to solve the problem *analytically*. Even in simple systems similar to the grocery with two assistants (Example 5.1), all the derived equations seem to be quite complex. Both the structure of a waiting line system and the parameters (e.g. type of the probability distribution used in the model) have significant impact on the complexity of solution. When the analytical approaches abort, *simulation experiments* are the only way how to treat the managerial problem.

Computer simulation can be defined as a special *method using computer experiments with the model of a real system*. Simulation approach can be successfully applied, of course, especially to complex waiting line models. First, we present a slight introduction into the terminology used in simulation models.

Entity is an object that goes through the model.
Resource is an agent required by the entity.

In the waiting line models entities correspond to the customers who enter the system in order to be served and resources are the servers providing the required service.

Event is a significant change of the system.
Activity is a process between two events.

Three basic types of events occur in the waiting line models: a customer's entry to the system, a start of the service and a finish of the service. Hence, two different activities run in such a model: waiting in the queue and the service.

If the behavior of a system is to be simulated, we must be able to *generate* the events in the system. For this purpose, it is necessary to know the probability distribution of all the random variables in the model. Thus, generated *random values* may correspond, for example, to the interarrival times or service times. S*imulation time* is the real time period throughout which the simulated system's performance is being monitored.

The most widespread *computer simulation languages* are GPSS/H, SLX, SIMSCRIPT, SLAM, SIMPLE++, AWESIM, etc. Simulation software has, as the basic component, a built-in statistics collection and printout of the results. In addition, cost analysis enables to optimize the system. Many software packages, especially the tutorial products, provide the graphical *animation* of the running simulation.

Since the computer simulation is just an experiment with the model, the obtained results differ from the values being calculated analytically (of course, in case the analytical solution exists at all). The difference between the analytical values and the values estimated from the simulation run become generally lower considering the longer simulation time.

Example 5.3

The simulation model has been developed for the example with two assistants in the grocery. Comparison of the results calculated analytically in Example 5.2 and the estimations obtained in the simulation run is presented in the following table (we have simulated one working month). It is evident that the computer simulation offers perfect approximation of the grocery's performance.

Characteristic	Analytical Results	Simulation
Utilization of the grocery	0.600	0.612
Average waiting time	3.8 min	3.8 min
Average waiting time in the queue	1.4 min	1.3 min
Average number of customers in the grocery	1.88	1.90
Average number of customers in the queue	0.68	0.67
Total monthly profit	330 000 CZK	333 250 CZK

5.4 Glossary

Arrival Rate – intenzita příchodu požadavků
The average number of customers arriving in the waiting line system per unit of time.

Computer Simulation – počítačová simulace; napodobení chování reálného systému
Numerical technique using a computer experiment with the model of a real system.

Customer – požadavek; zákazník
An object entering the system with its service requirements.

Interarrival Time – interval mezi příchody požadavků
The time between two consecutive arrivals.

Queue (Waiting Line) – fronta
One ore more customers waiting for service.

Server (Service Facility) – obslužné zařízení; obslužná linka
An object in the system providing service for entering customers.

Service Rate – intenzita obsluhy
The average number of customers being served in the waiting line system per unit of time.

Service Time – doba trvání obsluhy
The time required for completion of a service.

Source of Customers – zdroj požadavků
A group of potential customers who can enter the system.

Utilization of the System – intenzita provozu
The fraction of time the server is busy (computed as the ratio of the arrival rate to the service rate).

Waiting Line Models – modely front; modely hromadné obsluhy
Models describing systems with service facilities being required by arriving customers. Because of the server's limited capacity, queues occur in the system.

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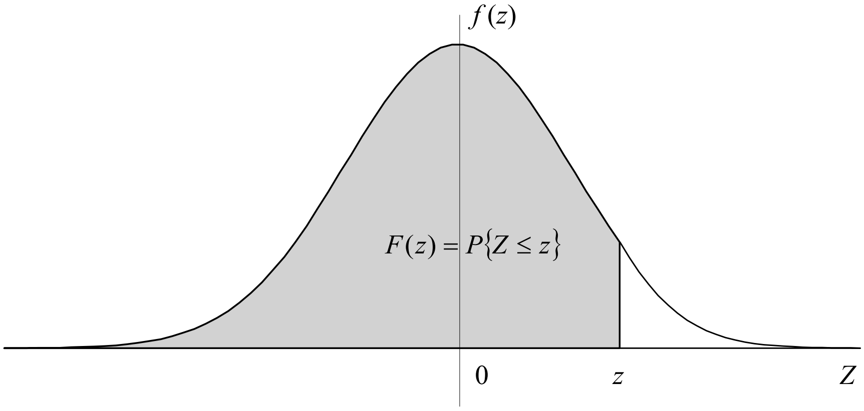
Appendix

Table of the standard normal distribution values



1. Determination of probability with which the standard normal distribution value Z is less than or equal to given value z.

   Examples:

   $z = -1.25 \quad \Rightarrow \quad \mathrm{P}\{Z \le -1.25\} = 0.10565.$

   $z = 1.82 \quad \Rightarrow \quad \mathrm{P}\{Z \le 1.82\} = 0.96562.$

2. Determination of value z for given probability of the standard normal distribution value $P\{Z \le z\}$.

   Examples:

   $P\{Z \le z\} = 0.261 \quad \Rightarrow \quad z = -0.64.$

   $P\{Z \le z\} = 0.614 \quad \Rightarrow \quad z = 0.29.$

Table of the standard normal distribution values ($z \le 0$)

$-z$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414
0.1	0.46017	0.45621	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42466
0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
0.3	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
0.4	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
0.5	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510
0.7	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
1.0	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
1.1	0.13567	0.13350	0.13136	0.12924	0.12714	0.12507	0.12302	0.12100	0.11900	0.11702
1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10384	0.10204	0.10027	0.09853
1.3	0.09680	0.09510	0.09342	0.09176	0.09012	0.08851	0.08692	0.08534	0.08379	0.08226
1.4	0.08076	0.07927	0.07780	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
1.6	0.05480	0.05370	0.05262	0.05155	0.05050	0.04947	0.04846	0.04746	0.04648	0.04551
1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.03920	0.03836	0.03754	0.03673
1.8	0.03593	0.03515	0.03438	0.03363	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
1.9	0.02872	0.02807	0.02743	0.02680	0.02619	0.02559	0.02500	0.02442	0.02385	0.02330
2.0	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.01970	0.01923	0.01876	0.01831
2.1	0.01786	0.01743	0.01700	0.01659	0.01618	0.01578	0.01539	0.01500	0.01463	0.01426
2.2	0.01390	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.01160	0.01130	0.01101
2.3	0.01072	0.01044	0.01017	0.00990	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
2.4	0.00820	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
2.5	0.00621	0.00604	0.00587	0.00570	0.00554	0.00539	0.00523	0.00509	0.00494	0.00480
2.6	0.00466	0.00453	0.00440	0.00427	0.00415	0.00403	0.00391	0.00379	0.00368	0.00357
2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.00280	0.00272	0.00264
2.8	0.00256	0.00248	0.00240	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
2.9	0.00187	0.00181	0.00175	0.00170	0.00164	0.00159	0.00154	0.00149	0.00144	0.00140
3.0	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.00100
3.1	0.00097	0.00094	0.00090	0.00087	0.00085	0.00082	0.00079	0.00076	0.00074	0.00071
3.2	0.00069	0.00066	0.00064	0.00062	0.00060	0.00058	0.00056	0.00054	0.00052	0.00050
3.3	0.00048	0.00047	0.00045	0.00043	0.00042	0.00040	0.00039	0.00038	0.00036	0.00035
3.4	0.00034	0.00033	0.00031	0.00030	0.00029	0.00028	0.00027	0.00026	0.00025	0.00024
3.5	0.00023	0.00022	0.00022	0.00021	0.00020	0.00019	0.00019	0.00018	0.00017	0.00017

Table of the standard normal distribution values ($z \geq 0$)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983